

Extension of Picard’s Iteration Method to a Class of n th-Order Initial Value Problems

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Abstract. This paper extends the method of successive approximations to an n th-order initial value problem without converting it to a first-order system. We obtain a bound in a closed form for the difference between two successive iterates. Finally, an error estimate for the solutions has been calculated and it is shown to be increasingly accurate as n increases. Suitable illustrations to support the result are provided.

Keywords: Gronwall’s inequality; Initial value problem; Integral equation; Lipschitz condition; Weierstrauss’ M-test.

1. Introduction

Differential equations together with the unknown functions and its derivatives at a given point in the domain constitute Initial Value Problems (IVPs). IVPs are used in modeling phenomena in physics, applied sciences, and engineering. While there are several available methods, mostly numerical, to compute solutions, it is noteworthy to mention certain developments in finding solutions to IVPs. A novel class of approximations—an approach explicit in nature for numerical

solutions of IVPs are constructed by Ramos et al. [12]. Some numerical methods have been used in [4] to analyze the solutions of IVPs by varying the length of the steps. Further, Islam [9] presented certain well-known numerical schemes for solving IVPs by investigating and computing errors between the Euler and Runge–Kutta methods. Erbay et al. [8] proved the local existence of strong solutions to the IVP used in modeling certain elasticity problems. With several approaches to obtain the solutions of IVPs, it is necessary to study the accuracy of numerical schemes. Akanbi [2] studied the popular Euler’s method for its accuracy by obtaining the error bounds. The numerical accuracy of a particular class of a specific growth model has been studied by Ju et al. [10]. An important application of error estimations can be found in [6], where the authors optimized the error estimates by using the K-fold cross validation method, while modeling the phenomena of global irradiation in Algerian deserts. Recently, the method of error estimations has been extended to various class of time-fractional differential equations. Li and Zhang [11] studied unconditionally optimized error estimates for a nonlinear time-fractional reaction-subdiffusion equation. Smadi and Arquib [13] developed an algorithm which is computational in nature to obtain error estimates in solving Fredholm integro-differential equations with time-fractional derivatives.

The Picard iteration method for optimal control problems with quadratic objective functional proposed by Akkouche and Aidene [3] was used to obtain approximate analytical solutions. At the same time, Bafghi et al. [5] developed an effective numerical iterative method, called the Jacobi–Picard iteration (JPI) method for nonlinear IVPs. Tisdell [14] excellently outlined the historical journey of the iteration method of Picard-type for differential equations and used to show the validity of Robin’s iterative scheme. It is noteworthy to mention that [15] deals with nonhomogeneous n th-order linear differential equations that have fuzzy initial conditions.

Motivated by the above mentioned work, in this paper, we extend the Picard method of successive approximations, found in [1, 7] for first-order IVPs of the type

$$y' = f(x, y), \quad y(x_0) = y_0$$

to an n th-order IVP of the type

$$y^{(n)} = f(x, y), \quad y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \dots, y^{(n-1)}(x_0) = y_{n-1},$$

without converting these to a system of first-order differential equations. Having calculated the first few terms and the difference between successive terms, a pattern is observed for this difference which is proved by induction. Finally, an error estimate for solutions has been calculated and an illustration has been provided.

2. Extension of Results to a Particular n th-Order IVP

In this section, we extend the Picard method of successive iterates for first-order

IVPs to the following class of IVPs:

$$y^{(n)}(x) = f(x, y), \quad x \in \mathcal{I}, \quad (1)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2, \dots, \quad y^{(n-1)}(x_0) = y_{n-1}, \quad (2)$$

where $f: \mathcal{I} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $\mathcal{I}$ is any closed interval in $\mathbb{R}$ and $x_0 \in \mathcal{I}$, $y_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$. The Leibnitz rule which is used in establishing our results is stated below.

Theorem 2.1. *If both $\Phi(x, y)$ and $\frac{\partial \Phi}{\partial x}$ are continuous in x and y , then for any two functions $\Psi_1(x)$ and $\Psi_2(x)$ which are continuously differentiable at least once, we have*

$$\frac{d}{dx} \int_{\Psi_1(x)}^{\Psi_2(x)} \Phi(x, y) dy = \int_{\Psi_1(x)}^{\Psi_2(x)} \frac{\partial \Phi}{\partial x} dy + \Phi[x, \Psi_2(x)] \frac{d\Psi_2(x)}{dx} - \Phi[x, \Psi_1(x)] \frac{d\Psi_1(x)}{dx}.$$

The following theorem establishes the equivalence between the class of IVPs (1)–(2) and the corresponding integral equation.

Theorem 2.2. *Let $f: \mathcal{I} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, where $\mathcal{I}$ is a closed interval in $\mathbb{R}$. Then every solution of the IVP (1)–(2) is given by*

$$\begin{aligned} y(x) = & y_0 + (x - x_0)y_1 + \frac{(x - x_0)^2}{2!}y_2 + \frac{(x - x_0)^3}{3!}y_3 \\ & + \dots + \int_{x_0}^x \frac{(x - t)^{n-1}}{(n-1)!} f(t, y(t)) dt. \end{aligned} \quad (3)$$

Proof. Differentiating (1) with respect to x and using (2), we get

$$y^{(n-1)}(x) = y_{n-1} + \int_{x_0}^x f(t, y(t)) dt.$$

On differentiating this equation with respect to x and using (2), we obtain

$$y^{(n-2)}(x) = y_{n-2} + (x - x_0)y_{n-1} + \int_{x_0}^x f(t, y(t)) dt.$$

Now, continuing this procedure another $(n-2)$ -times, we get

$$\begin{aligned} y(x) = & y_0 + (x - x_0)y_1 + \frac{(x - x_0)^2}{2!}y_2 + \frac{(x - x_0)^3}{3!}y_3 + \dots \\ & + \int_{x_0}^x \int_{x_0}^{t_n} \dots \int_{x_0}^{t_3} \int_{x_0}^{t_2} f(t_1, y(t_1)) dt_1 dt_2 \dots dt_{n-1} dt_n. \end{aligned} \quad (4)$$

We shall convert the multiple integral in (4) to a single integral. To do this, consider

$$I_n(x) := \int_{x_0}^x (x-t)^{n-1} y'(t) dt, \quad (5)$$

where $n \in \mathbb{N}$ and $x_0 \in \mathbb{R}$ is a constant.

Differentiating (5) with respect to x and using the Leibnitz rule, we obtain

$$\begin{aligned} \frac{dI_n(x)}{dx} &= \int_{x_0}^x \frac{\partial}{\partial x} (x-t)^{n-1} f(t, y(t)) dt + (x-x)^{n-1} f(x, y(x)) \frac{dx}{dx} \\ &\quad - (x-x_0)^{n-1} f(x_0, y(x_0)) \frac{dx_0}{dx}. \end{aligned}$$

This yields

$$\frac{dI_n(x)}{dx} = (n-1) \int_{x_0}^x (x-t)^{n-2} f(t, y(t)) dt = (n-1) I_{n-1}(x), \quad n > 1. \quad (6)$$

From (5), we have

$$I_1(x) = \int_{x_0}^x f(t, y(t)) dt$$

and then

$$\frac{dI_1(x)}{dx} = f(x, y(x)). \quad (7)$$

Differentiating (6) with respect to x successively k -times and using the Leibnitz rule, we obtain

$$\frac{d^k I_n(x)}{dx^k} = (n-1)(n-2) \cdots (n-k) I_{n-k}(x), \quad n > k. \quad (8)$$

For $k = n-1$, (8) yields

$$\frac{d^{n-1} I_n(x)}{dx^{n-1}} = (n-1)! I_1(x). \quad (9)$$

Again, differentiate (9) with respect to x and use (7), we obtain

$$\frac{d^n I_n(x)}{dx^n} = (n-1)! f(x, y(x)). \quad (10)$$

From (5), (8), and (9), we see that the first $(n-1)$ derivatives of $I_n(x)$ vanish at $x = x_0$. Hence, using (7) and (10), we obtain

$$I_1(x) = \int_{x_0}^x f(t_1, y(t_1)) dt_1,$$

$$I_2(x) = \int_{x_0}^x I_1(t_2) dt_2 = \int_{x_0}^x \int_{x_0}^{t_2} f(t_1, y(t_1)) dt_1 dt_2.$$

Proceeding likewise, we obtain

$$I_n(x) = (n-1)! \int_{x_0}^x \int_{x_0}^{t_n} \cdots \int_{x_0}^{t_3} \int_{x_0}^{t_2} f(t_1, y(t_1)) dt_1 dt_2 \cdots dt_{n-1} dt_n. \quad (11)$$

Combining (5) and (11), we get

$$\begin{aligned} & \int_{x_0}^x \int_{x_0}^{t_n} \cdots \int_{x_0}^{t_3} \int_{x_0}^{t_2} f(t_1, y(t_1)) dt_1 dt_2 \cdots dt_{n-1} dt_n \\ &= \frac{1}{(n-1)!} \int_{x_0}^x (x-t)^{n-1} f(t, y(t)) dt. \end{aligned} \quad (12)$$

Using (12) in (4), we get (3). This completes the proof. $\blacksquare$

In view of Theorem 2.2, we define the Picard iterative scheme for the IVP (1)–(2) by

$$\begin{aligned} y_{m+1}(x) &= y_0 + (x-x_0)y_1 + \frac{(x-x_0)^2}{2!}y_2 + \frac{(x-x_0)^3}{3!}y_3 + \cdots \\ &+ \int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} f(t, y_m(t)) dt, \end{aligned} \quad (13)$$

where $x \in \mathcal{I}$ and $m \in \mathbb{N}_0$.

Theorem 2.3. Let $\overline{S} := \{(x, y) \in \mathbb{R}^2 : |x-x_0| \leq a, |y-y_0| \leq b \text{ for some } a, b > 0\}$ and $f: \overline{S} \rightarrow \mathbb{R}$ be a continuous function. Assume that there exists $M > 0$ such that

$$|f(x, y)| \leq M \quad \text{for all } (x, y) \in \overline{S},$$

and there exists $L > 0$ such that

$$|f(x, y) - f(x, z)| \leq L|y - z| \quad \text{for all } (x, y), (x, z) \in \overline{S}.$$

If y_0 is a continuous function on $\mathcal{J} := \{x \in \mathbb{R} : |x-x_0| \leq a\}$, then there exists a sequence of functions $\{y_m\}$, generated by (13), converging to the solution y of (1)–(2) valid in $\mathcal{J}_h := \{x \in \mathbb{R} : |x-x_0| \leq h\}$, where $h := \min \left\{ a, \frac{b}{M_1} \right\}$, where $M_1 = |y_1| + |y_2| \frac{a}{2!} + |y_3| \frac{a^2}{3!} + \cdots + \frac{M a^{n-2}}{(n-1)!}$.

Proof. Consider the approximations, defined by (13), each of which are continuous on $\mathcal{J}_h$. Then for $m = 0$,

$$\begin{aligned}
 |y_1(x) - y_0| &= \left| y_0 + (x - x_0)y_1 + \frac{(x - x_0)^2}{2!}y_2 + \frac{(x - x_0)^3}{3!}y_3 + \cdots \right. \\
 &\quad \left. + \int_{x_0}^x \frac{(x - t)^{n-1}}{(n-1)!} f(t, y_0(t)) dt - y_0 \right| \\
 &\leq |y_1||h| + |y_2|\frac{|h|^2}{2!} + |y_3|\frac{|h|^3}{3!} + \cdots + \frac{M|h|^{n-1}}{(n-1)!} \\
 &\leq h \left[|y_1| + |y_2|\frac{a}{2!} + |y_3|\frac{a^2}{3!} + \cdots + \frac{Ma^{n-2}}{(n-1)!} \right] \\
 &\leq hM_1 \\
 &\leq b.
 \end{aligned}$$

Now, assuming that $|y_{m-1}(x) - y_0| \leq b$ for all $m \geq 1$, we claim $|y_m(x) - y_0| \leq b$. The continuity of $f(x, y_{m-1}(x))$ on $\mathcal{J}_h$ follows from that of y_{m-1} on $\mathcal{J}_h$. Further,

$$\begin{aligned}
 |y_m(x) - y_0| &= \left| y_0 + (x - x_0)y_1 + \frac{(x - x_0)^2}{2!}y_2 + \frac{(x - x_0)^3}{3!}y_3 + \cdots \right. \\
 &\quad \left. + \int_{x_0}^x \frac{(x - t)^{n-1}}{(n-1)!} f(t, y_{m-1}(t)) dt - y_0 \right| \\
 &\leq |y_1|h + |y_2|\frac{|h|^2}{2!} + |y_3|\frac{|h|^3}{3!} + \cdots + \frac{Mh^{n-1}}{(n-1)!} \\
 &\leq h \left[|y_1| + |y_2|\frac{a}{2!} + |y_3|\frac{a^2}{3!} + \cdots + \frac{Ma^{n-2}}{(n-1)!} \right] \\
 &\leq hM_1 \\
 &\leq b.
 \end{aligned}$$

Since y_0 and y_1 are continuous on a compact interval $\mathcal{J}_h$, there exists a positive real N such that $\sup_{x \in \mathcal{J}_h} |y_1(x) - y_0(x)| \leq N$.

We now establish a bound for the difference between two successive iterates, namely

$$|y_{m+1}(x) - y_m(x)| \leq N \frac{L^m |x - x_0|^{nm}}{(nm)!}, \quad m \in \mathbb{N}_0. \quad (14)$$

We shall prove (14) by the principle of mathematical induction. The case for $m = 0$ is argued earlier.

We assume (14) for $m = k \geq 2$, i.e., $|y_{k+1}(x) - y_k(x)| \leq N \frac{L^k |x - x_0|^{nk}}{(nk)!}$, and will prove it for $m = k + 1$, i.e., $|y_{k+2}(x) - y_{k+1}(x)| \leq N \frac{L^{k+1} |x - x_0|^{n(k+1)}}{(n(k+1))!}$, $k = 0, 1, 2, \dots$.

The main ideas in establishing a bound for the difference between successive iterates begins by substitution of the values of the solutions y_{k+2} and y_{k+1} . Further, by hypothesis $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ on $\bar{S}$. The Lipschitz condition on f then brings us to the induction hypothesis step. The mathematical induction is then completed by using the binomial theorem twice. Now,

$$\begin{aligned}
& |y_{k+2}(x) - y_{k+1}(x)| \\
&= \left| \left(y_0 + (x - x_0)y_1 + \frac{(x - x_0)^2}{2!}y_2 + \cdots + \int_{x_0}^x \frac{(x - t)^{n-1}}{(n-1)!} f(t, y_{k+1}(t)) dt \right) \right. \\
&\quad \left. - \left(y_0 + (x - x_0)y_1 + \frac{(x - x_0)^2}{2!}y_2 + \cdots + \int_{x_0}^x \frac{(x - t)^{n-1}}{(n-1)!} f(t, y_k(t)) dt \right) \right| \\
&\leq L \int_{x_0}^x \left| \frac{(x - t)^{n-1}}{(n-1)!} \right| |y_{k+1}(t) - y_k(t)| dt \\
&\leq \frac{NL^{k+1}}{(nk)!(n-1)!} \int_{x_0}^x |(x - t)^{n-1}| |t - x_0|^{nk} dt \\
&= \frac{NL^{k+1}}{(nk)!(n-1)!} \int_{x_0}^x \left| x^{n-1} - \binom{n-1}{1} x^{n-2}t + \binom{n-1}{2} x^{n-3}t^2 - \cdots \right. \\
&\quad \left. + (-1)^{n-1} t^{n-1} \right| \left| t^{2k} - \binom{2k}{1} t^{2k-1}x_0 + \binom{2k}{2} t^{2k-2}x_0^2 - \cdots + x_0^{2k} \right| dt \\
&\leq \frac{NL^{k+1}}{(nk)!(nk+1)(nk+2) \cdots (nk+n)(n-1)!} \\
&\quad \times ((n-1)!x^{nk+n} - n!(k+1)x^{nk+n-1}x_0 \\
&\quad + \frac{n!(nk+n-1)}{2}x^{nk+n-2}x_0^2 - \frac{n!(nk+n-1)(nk+n-2)}{3!}x^{nk+n-3}x_0^3 \\
&\quad + \cdots + (-1)^{nk+n}(n-1)!x_0^{nk+n}) \\
&\leq N \frac{L^{k+1}|x - x_0|^{n(k+1)}}{(nk+n)!}.
\end{aligned}$$

By the principle of mathematical induction, we have established (14) for all $m \in \mathbb{N}_0$ and $x \in \mathcal{J}_h$. Next,

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{NL^m|x - x_0|^{nm}}{(nm)!} &= N \sum_{m=0}^{\infty} \frac{(\sqrt[n]{L}|x - x_0|)^{nm}}{(nm)!} \\
&\leq N \sum_{m=0}^{\infty} \frac{(\sqrt[n]{L}h)^{nm}}{(nm)!}
\end{aligned}$$

$$= N \exp(\sqrt[n]{L}h) \\ < \infty.$$

By the Weierstrass M-test, $\sum_{m=0}^{\infty} (y_{m+1}(x) - y_m(x))$ converges uniformly and absolutely on $\mathcal{J}_h$ to a continuous function y on $\mathcal{J}_h$. This y solves the integral equation (3). The uniqueness of solution easily follows from Gronwall's inequality, see [1, 7]. For if z is another solution of (3) on $\mathcal{J}_h$, and $(x, z(x)) \in \bar{S}$, then

$$\begin{aligned} |y(x) - z(x)| &= \left| \left(y_0 + (x - x_0)y_1 + \frac{(x - x_0)^2}{2!}y_2 + \frac{(x - x_0)^3}{3!}y_3 + \cdots \right. \right. \\ &\quad \left. \left. + \int_{x_0}^x \frac{(x - t)^{n-1}}{(n-1)!} f(t, y(t)) dt \right) \right. \\ &\quad \left. - \left(y_0 + (x - x_0)y_1 + \frac{(x - x_0)^2}{2!}y_2 + \frac{(x - x_0)^3}{3!}y_3 + \cdots \right. \right. \\ &\quad \left. \left. + \int_{x_0}^x \frac{(x - t)^{n-1}}{(n-1)!} f(t, z(t)) dt \right) \right| \\ &\leq \int_{x_0}^x \frac{(x - t)^{n-1}}{(n-1)!} |f(t, y(t)) - f(t, z(t))| dt. \end{aligned}$$

By hypothesis, we can write

$$\begin{aligned} |y(x) - z(x)| &\leq \frac{L}{(n-1)!} \int_{x_0}^x (x - t)^{n-1} |y(t) - z(t)| dt \\ &< \varepsilon + \frac{L}{(n-1)!} \int_{x_0}^x |x - t|^{n-1} |y(t) - z(t)| dt \quad \text{for some } \varepsilon > 0. \end{aligned}$$

Applying Gronwall's inequality, we obtain

$$|y(x) - z(x)| \leq \varepsilon \exp \left(\left| \int_{x_0}^x \frac{L|x - t|^{n-1}}{(n-1)!} dt \right| \right).$$

Since $\varepsilon > 0$ is arbitrary, the uniqueness of solution of (3) follows. ■

Theorem 2.4. *Suppose that the hypotheses of Theorem 2.3 hold. Then for all $x \in \mathcal{J}_h$, the error estimate is computed by the following expression.*

$$|y(x) - y_m(x)| \leq N \exp(\sqrt[n]{L}h) \min \left\{ 1, \frac{(\sqrt[n]{L}h)^{nm}}{(nm)!} \right\}, \quad m = 0, 1, 2, \dots, \quad (15)$$

where $\max_{x \in \mathcal{J}_h} |y_1(x) - y_0(x)| \leq N$.

Proof. For $l > m$, the triangle inequality gives,

$$\begin{aligned}
& |y_l(x) - y_m(x)| \\
& \leq \sum_{k=m}^{l-1} |y_{k+1}(x) - y_k(x)| \\
& \stackrel{(14)}{\leq} \sum_{k=m}^{l-1} NL^k \frac{|x - x_0|^{nk}}{(nk)!} \\
& \leq N \left[\frac{(\sqrt[n]{L}h)^{nm}}{(nm)!} + \frac{(\sqrt[n]{L}h)^{nm+n}}{(nm+n)!} + \frac{(\sqrt[n]{L}h)^{nm+2n}}{(nm+2n)!} + \cdots + \frac{(\sqrt[n]{L}h)^{ln-n}}{(ln-n)!} \right] \\
& \leq \frac{N(\sqrt[n]{L}h)^{mn}}{(mn)!} \left[1 + \frac{(\sqrt[n]{L}h)^n}{(nm+n)(nm+n-1)\cdots(nm+1)} \right. \\
& \quad + \frac{(\sqrt[n]{L}h)^{2n}}{(nm+2n)(nm+2n-1)\cdots(nm+1)} + \cdots \\
& \quad \left. + \frac{(\sqrt[n]{L}h)^{nl-nm-1}}{(nl-n)(nl-n-1)\cdots(nm+1)} \right].
\end{aligned}$$

Since each term in the bracket above is positive, we can write

$$|y_l(x) - y_m(x)| \leq \frac{N(\sqrt[n]{L}h)^{nm}}{(nm)!} \sum_{k=0}^{\infty} \frac{(\sqrt[n]{L}h)^k}{k!} = \frac{N(\sqrt[n]{L}h)^{mn}}{(mn)!} \exp(\sqrt[n]{L}h).$$

Letting l approaches to ∞ , and by noting that $\lim_{l \rightarrow \infty} y_n(x) = y(x)$,

$$|y(x) - y_m(x)| \leq N \frac{(\sqrt[n]{L}h)^{nm}}{(nm)!} \exp(\sqrt[n]{L}h). \quad (16)$$

Further, we already have $|y_n(x) - y_m(x)| \leq N \exp(\sqrt[n]{L}h)$. Letting n tends to ∞ , we obtain

$$|y(x) - y_m(x)| \leq N \exp(\sqrt[n]{L}h). \quad (17)$$

From (16) and (17), we obtain the desired error bound (15). $\blacksquare$

3. Some Illustrative Examples

Example 3.1. Let $\bar{S} := \{(x, y) \in \mathbb{R}^2 : |x| \leq \frac{1}{4}, |y - 1| \leq 1\}$ and consider the first-order IVP

$$y'(x) = y(x), \quad y(0) = 1. \quad (18)$$

The exact solution of (18) is $y(x) = e^x$. The first three iterates by Picard's method are $y_1(x) = 1 + x$, $y_2(x) = 1 + x + \frac{x^2}{2!}$, and $c(x) := y_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 0.00017125 + K \quad \text{where} \quad K := \sum_{i=10}^{\infty} \frac{1}{4^i i!}. \quad (19)$$

Next, consider the second-order IVP

$$y''(x) = y(x), \quad y(0) = 1, \quad y'(0) = 1. \quad (20)$$

The exact solution of (20) is also $y(x) = e^x$. The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= 1 + x + \frac{x^2}{2}, \\ y_2(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}, \\ b(x) := y_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 0.000163 + K \quad \text{where} \quad K := \sum_{i=10}^{\infty} \frac{1}{4^i i!}. \quad (21)$$

Finally, consider the third-order IVP

$$y'''(x) = y(x), \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 1. \quad (22)$$

The exact solution of (22) is also $y(x) = e^x$. The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, \\ y_2(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}, \\ a(x) := y_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880}. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq K \quad \text{where} \quad K := \sum_{i=10}^{\infty} \frac{1}{4^i i!}. \quad (23)$$

It can be seen from (19), (21) and (23), the error estimates are tightest for IVP (22), followed by IVP (20) and (18). In Figure 1, the green colored curve represents the exact solution. The blue, brown and red colored curves indicate the third approximation of the third, second, and first-order IVP respectively. The graph illustrates the increase in the convergence with the increase in the order of the IVP.

As a next illustration, consider another example.

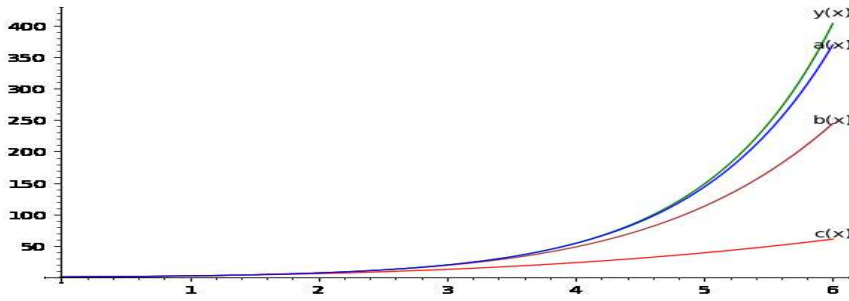


Figure 1: Comparison among the third approximations of different derivatives

Example 3.2. Let $\overline{S}_1 := \{(x, y) \in \mathbb{R}^2 : |x| \leq 4, |y - 1| \leq 5\}$ and consider the first-order IVP

$$y'(x) = \sin x + y(x), \quad y(0) = 1. \quad (24)$$

The exact solution of (24) is computed to be $y(x) = \frac{1}{2}(3e^x - \cos x - \sin x)$. The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= 2 + x - \cos x, \\ y_2(x) &= 2 + 2x + \frac{1}{2}(x^2 - 2\cos x - 2\sin x), \\ y_3(x) &= 1 + 2x + x^2 + \frac{x^3}{6} - \sin x. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 46.1789. \quad (25)$$

The percentage error in this case is 55.9%. Next, consider the second-order IVP

$$y''(x) = \sin x + y(x), \quad y(0) = 1, \quad y'(0) = 1. \quad (26)$$

The exact solution of (26) is found to be $y(x) = \frac{1}{4}e^{-x}(5e^{2x} - 2e^x \sin x - 1)$. The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= 1 + x + \frac{1}{2}x(x+2) - \sin x, \\ y_2(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{24}, \\ y_3(x) &= 1 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{60} + \frac{x^6}{720} - \sin x. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 6.7758. \quad (27)$$

The percentage error in this case is 9.9%.

Finally, consider the third-order IVP

$$y'''(x) = \sin x + y(x), \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 1. \quad (28)$$

The exact solution of (28) is found to be

$$\begin{aligned} y(x) = & \frac{1}{6}e^{-x/2} \left(7e^{3x/2} - e^{x/2} \cos x - 4 \cos(\sqrt{3}x/2) \right. \\ & + 4e^{x/2} \cos x \cos^2(\sqrt{3}x/2) - e^{x/2} \sin x - 2e^{x/2} \cos^2(\sqrt{3}x/2) \sin x \\ & \left. + 4e^{x/2} \cos x \sin^2(\sqrt{3}x/2) - 2e^{x/2} \sin x \sin^2(\sqrt{3}x/2) \right). \end{aligned}$$

The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= x + x^2 + \frac{x^3}{6} + \cos x, \\ y_2(x) &= 2x + x^2 + \frac{x^4}{24} + \frac{x^5}{60} + \frac{x^6}{720} + \cos x - \sin x, \\ y_3(x) &= 1 + 2x + \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^7}{5040} + \frac{x^9}{362880} - \sin x. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 3.705. \quad (29)$$

The percentage error in this case is 5.8%. Comparing (25), (27), and (29), we see that there is a massive reduction in the percentage error and thus we get a tighter error bound meaning a better approximate for the solution of the third-order IVP.

In Figure 2, the curves corresponding to the first-order IVP appear in red color. The exact solution is labeled $p(x)$ and the third approximation using Picard's scheme is denoted by $u(x)$. The curves in blue color represent the IVP of second-order and indicate the exact and approximate solution. The labels $q(x)$ and $v(x)$ denote the exact and third approximation. Finally, the green colored curves (inside the blue colored curves) marked as $r(x)$ and $w(x)$ to denote the exact and third approximation represent the third-order IVP. As the gap between the red colored curves is the widest while the gap between the green colored curves is the narrowest, it indeed can be seen that the convergence rate increases as the order of the derivative increases.

4. Conclusion

We have extended Picard's method of successive approximations to a particular type of an n th-order initial value problem and obtained a compact form for the difference between two successive approximate solutions. Further, we have obtained a bound for the error using this method. For the class of the n th-order

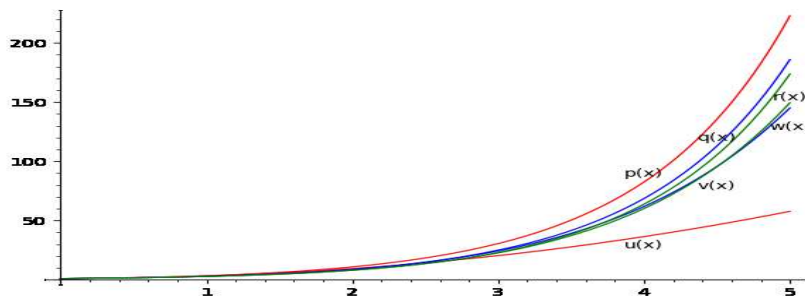


Figure 2: Comparison between the exact solutions and third approximations of the different order IVPs

initial value problem considered, the percentage error starts dropping significantly, as n starts increasing. This means that the successive iterates obtained by Picard's method, converge rapidly to the exact solution as n increases.

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References

- [1] R.P. Agarwal, D. O'Regan, *An Introduction to Ordinary Differential Equations*, Springer, New York, 2012.
- [2] A.M. Akanbi, Propagation of errors in Eulers method, *Arch. Appl. Sci. Res.* **2** (2) (2010) 457–469.
- [3] A. Akkouché, M. Aidene, Solving optimal control problems using the Picard's iteration method, *RAIRO-Oper. Res.* **54** (5) (2020) 1419–1435.
- [4] M.A. Arefin, B. Gain, R. Karim, Accuracy analysis on solution of initial value problems of ordinary differential equations for some numerical methods with different step sizes, *Int. Ann. Sci.* **10** (1) (2021) 118–133.
- [5] M.T. Bafghi, G.B. Loghmani, M. Heydari, X. Bai, Jacobi–Picard iteration method for the numerical solution of nonlinear initial value problems, *Math. Methods Appl. Sci.* **43** (3) (2020) 1084–1111.
- [6] K. Bouchouicha, M.A. Hassan, N. Bailek, N. Aoun, Estimating the global solar irradiation and optimizing the error estimates under algerian desert climate, *Renew. Energ.* **139** (2019) 844–858.
- [7] S.G. Deo, V. Raghavendra, R. Kar, V. Lakshmikantham, *Textbook of Ordinary Differential Equations*, McGraw Hill Education, India, 2015.
- [8] H.A. Erbay, A. Erkip, Y. Sengul, Local existence of solutions to the initial value problem for one-dimensional strain limiting viscoelasticity, *J. Diff. Equ.* **269** (11) (2020) 9720–9739.
- [9] A. Islam, A comparative study on numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE) with Euler and Runge Kutta methods, *Am. J. Comput. Math.* **5** (3) (2015) 393–404.
- [10] L. Ju, X. Li, Z. Qiao, H. Zhang, Energy stability and error estimates of exponential time differencing schemes for the Epitaxial growth model without slope selection, *Math. Comput.* **87** (2018) 1859–1885.

- [11] D. Li, J. Zhang, Z. Zhang, Unconditionally optimal error estimates of a linearized Galerkin method for nonlinear time-fractional reaction-subdiffusion equations, *J. Sci. Comput.* **76** (2018) 848–866.
- [12] H. Ramos, G. Singh, V. Kanwar, S. Bhatia, Solving first-order initial value problems by using an explicit non-standard A-stable one step method in variable step size formulation, *Appl. Math. Comput.* **268** (1) (2015) 796–805.
- [13] M.A. Smadi, O.A. Arqub, Computational algorithm for solving fredholm time fractional partial integrodifferential equations of Dirichlet functions type with error estimates, *Appl. Math. Comput.* **342** (1) (2019) 280–294.
- [14] C.C. Tisdell, On Picard’s iteration method to solve differential equations and a Pedagogical space for otherness, *Int. J. Math. Educ. Sci. Technol.* **50** (5) (2019) 788–799.
- [15] V.F. Wasques, F.E. Laureano, L.C. de Barros, F.S. Pedro, P. Sussner, Higher order initial value problem with interactive fuzzy conditions, In: *IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, 2018.
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