



Existence and stability of solutions of differential equations involving generalized piecewise constant arguments

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Abstract

This paper focuses on proving the existence of solutions for differential equations with a generalized piecewise constant argument by employing absolutely continuous functions. Accordingly, the concept of absolutely continuous solutions for differential equations with a piecewise constant argument of generalized type (DEPCAG) is introduced. Furthermore, by utilizing nonsmooth Lyapunov functions, Lyapunov's second method is extended to the DEPCAG framework. One of the theorems developed herein is then applied to analyze the stability of a specific class of DEPCAG. The applicability of the results established is illustrated through a suitable example of the logistic growth model.

Keywords Nonlinear differential equations · Piecewise constant argument of generalized type · Existence · Lyapunov stability

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1 Introduction

The conventional ordinary differential equations are effective in modeling systems with smooth dynamics. However, there are many systems which have nonsmooth dynamics. The systems with nonsmooth dynamics cannot be studied by means of merely ordinary differential equations. The presence of discontinuity necessitates the need to develop the theory of differential equations with discontinuities. There exist various forms of differential equations that incorporate discontinuities, among which differential equations with piecewise constant argument (DEPCA) form a well known category. The investigation into DEPCAs began approximately four decades ago with the work of Busenberg and Cooke [12], who introduced a mathematical model featuring such deviating arguments in the context of vertically transmitted disease dynamics. Right after this, Cooke and Wiener [18] and Shah and Wiener [29] studied different features of DEPCA. Since then, a lot of investigation has been done on diversified aspects of the DEPCA. To be specific, Cooke and Wiener [17] surveyed various type of DEPCA including linear and nonlinear, retarded, advanced, neutral, and mixed. Results concerning the existence, uniqueness, representation, and stability of these types of equations are also derived in that paper. Sakata and Hara [28] discussed dynamics of a linear DEPCA. Küpper and Yuang [23] explored the existence of quasi-periodic solutions for DEPCA

$$\dot{x}(t) = ax(t) + \sum_{i=-N}^N a_i x([t+i]) + f(t), \quad N \geq 1,$$

where f is a periodic function with period $T \in \mathbb{R} \setminus \mathbb{Q}$. It was demonstrated that DEPCAs subjected to periodic perturbations admit quasi-periodic solutions, yet do not possess any periodic solutions.

Cabada et al. [14] constructed Green's function and established comparison principles for the first-order periodic DEPCA

$$\dot{x}(t) + mx(t) + Mx([t]) = f(t), \quad t \in [0, T], \quad x(0) = x(T) + \lambda,$$

where m, M, T , and λ are real constants with $T > 0$, and f is a piecewise continuous function on $[0, T]$.

The initial value problem below was first analyzed by Cabada and Ferreiro [13]

$$\dot{x}(t) + mx(t) + Mx([t]) = f(t), \quad t \in [0, T], \quad x(0) = \lambda,$$

for the uniqueness of its solution, from which they derived comparison results. They provided results on the existence of extremal quasi-solutions and the uniqueness of solutions when weakly coupled lower and upper solutions are present. Additionally, they demonstrated the existence of unique extremal solutions situated between a pair of well-ordered lower and upper solutions for the DEPCA

$$\dot{x}(t) = f(t, x(t), x([t])), \quad t \in [0, T], \quad g(x(0), x(T)) = 0,$$

where T is positive real number and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Ivanov [21] investigated various aspects of the global dynamics of DEPCAs, including invariance, global stability of equilibria, existence, stability, and instability of solutions, as well as the structure of periodic solutions and the emergence of chaotic behavior in a DEPCA of the form

$$\varepsilon \dot{x}(t) + x(t) = f(x[t]), \quad t \in \mathbb{R}^+,$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and the parameter ε is such that $0 < \varepsilon < 1$, which is actually a special case of the DEPCA

$$\varepsilon \dot{x}(t) + x(t) = f(x[t] - k), \quad k \in \mathbb{N}_0.$$

Certain dynamic properties of this equation, such as invariance, existence, stability, and the structure of periodic solutions, are further analyzed by Ivanov and Trofimchuk [22].

Over time, considerable focus has been directed towards exploring the diverse characteristics of DEPCAs. Traditionally, however, the deviating argument in DEPCAs is defined using the greatest integer function, which remains constant over specific intervals. Yet, this assumption does not always align with real-world scenarios. To address this limitation, Akhmet [3] introduced a generalized version of DEPCA by substituting the greatest integer function with an arbitrary piecewise constant function. This extension, known as differential equations with piecewise constant argument of generalized type (DEPCAG), offers a more versatile framework and proves effective in modeling and analyzing a wide range of problems in discontinuous dynamical systems, hybrid systems, and related fields.

Akhmet [3] proved the existence of global solutions for quasilinear DEPCAGs under the condition that the corresponding linear homogeneous system exhibits an exponential dichotomy. Utilizing a novel approach grounded in an integral representation formula, Akhmet further examined the existence of bounded and periodic solutions. The work in [3] is mainly concerned with the quasilinear DEPCAG

$$\dot{x}(t) = A(t)x(t) + f(t, x(t), x(\beta(t))), \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad (1)$$

where $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$, and $\{\theta_i\}$ is a strictly monotone sequence of unbounded reals. Further, there exists $\theta > 0$ such that $\theta_{i+1} - \theta_i \leq \theta$, and $A : \mathbb{R} \rightarrow \mathbb{R}^n$ is continuous. As the investigative approach relies on formulating an equivalent integral equation, it eliminates the need to impose conditions on the corresponding reduced discrete equations. Consequently, the method requires conditions that are more straightforward to verify, akin to those used in ordinary differential equations.

Later on, using the same method of investigation, Akhmet [2] studied various properties, like the existence and stability of almost periodic solutions of hybrid DEPCAG

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + f(t, x(t), x(\theta_{\beta(t)-p_1}), x(\theta_{\beta(t)-p_2}), \dots, x(\theta_{\beta(t)-p_m})), \\ x(t) &\in \mathbb{R}^n, \quad t \in \mathbb{R}, \end{aligned}$$

where $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$, and $\{\theta_i\}$ is a strictly monotone sequence of unbounded reals, p_j , $j \in \{1, 2, \dots, m\}$, are fixed, and $A : \mathbb{R} \rightarrow \mathbb{R}^n$ is continuous. Moreover, Akhmet and Büyükdalı [7] investigated the criteria for periodic solutions to exist and parameter dependence of such solutions of the following DEPCAG

$$\dot{x}(t) = A(t)x(t) + f(t) + \mu g(t, x(t), x(\beta(t)), \mu), \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

where $\beta(t) = \xi_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$, and $\{\theta_i\}$ is a strictly monotone sequence of unbounded reals, $\xi_i \in [\theta_i, \theta_{i+1}]$, $\mu \in J$ with J as an open interval in $\mathbb{R}^n$ containing 0, and the functions $A : \mathbb{R} \rightarrow \mathbb{R}^n$, $f : J \rightarrow \mathbb{R}$, and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are all continuous.

As a continuation of [3], the sets of solutions' structure was specified by Akhmet [4] who obtained necessary and sufficient conditions for the zero solution, to the linear and quasilinear DEPCAG of the following type, to be stable:

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(\beta(t)), \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

and

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(\beta(t)) + f(t, x(t), x(\beta(t))), \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

where $\beta(t) = \xi_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$, and $\{\theta_i\}$ is a strictly monotone sequence of unbounded reals, $\xi_i \in [\theta_i, \theta_{i+1}]$, and $A_0, A_1 : \mathbb{R} \rightarrow \mathbb{R}^n$. Further, the continuity of $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are assumed. Various criteria for the global exponential stability of recurrent neural networks modeled by DEPCAGs have been established by Akhmet et al. [5], employing the Lyapunov function method.

Under specific integrability conditions, by utilizing a Gronwall–Bellman type inequality along with the Banach fixed point theorem, Castilo et al. [15] derived asymptotic formulae for the impulsive DEPCAGs

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(\beta(t))), \quad \text{for } t \neq t_i, \\ \Delta x(t_i) &= Q_i(x(t_i)), \quad \text{for } t = t_i, i \in \mathbb{N}, \\ x(\tau) &= x_0, \end{aligned}$$

where $\beta(t) = \xi_i$ if $t \in [t_i, t_{i+1})$, $i \in \mathbb{Z}$, and $\{t_i\}$ is a strictly monotone sequence of reals unbounded on the left as well as on the right, $\xi_i \in [t_i, t_{i+1}]$, and $Q_i : \{t_i\} \rightarrow \mathbb{R}^n$. The existence of an asymptotic equilibrium for the class linear and semilinear impulsive DEPCAG have been studied.

Chiu and Pinto [16], by using fixed point theorems and a new analysis technique, obtained conditions for the periodic solutions of a quasilinear DEPCAG (1) to exist.

In recent work, Abildayeva et al. [1] investigated a multi-point boundary value problem for a system of DEPCAGs by transforming it into a Cauchy problem on subintervals for linear systems of ordinary differential equations with parameters. Similarly, Assanova and Uteshova [9] developed methods to address two-point boundary value problems for DEPCAG systems by converting them into systems of Cauchy problems with parameters defined over subintervals of the partition. It is also interesting to read on the extension of Chaplygin's method to second-order differential equations of neutral type with piecewise

constant arguments by Lobo et. al. [24] to construct upper and lower solutions converging to the exact solution of this DEPCA.

In recent years, the investigation of absolutely continuous (AC) solutions to differential equations has garnered growing interest among researchers. For instance, Topolski [32] employed upper and lower absolutely continuous functions as subsolutions and supersolutions for discontinuous ordinary differential equations, providing sufficient conditions for the existence of extremal solutions to initial value problems. Pouso and Rodríguez-López [27] contributed new findings on the existence and uniqueness of absolutely continuous solutions for systems of discontinuous ordinary differential equations, assuming localized Bressan–Shen transversality conditions at the interfaces between distinct Lipschitz continuity domains. The use of absolutely continuous functions proves to be particularly effective and advantageous in the analysis of discontinuous differential systems, especially when exploring the existence and stability of equilibria and examining their qualitative dynamics, due to their inherent ability to accommodate discontinuities.

Motivated by the above mentioned work, two issues are addressed in this paper, namely, the existence of absolutely continuous solutions and the Lyapunov stability of the nonlinear DEPCAG

$$\dot{x}(t) = f(t, x(t), x(\beta(t))), \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \quad (2)$$

where $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$, and $\{\theta_i\}$ is a strictly increasing unbounded sequence in $\mathbb{R}$, and $f: \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given function, possibly nonlinear.

We establish the existence of absolutely continuous solutions employing the constructive approach and the stability employing Lyapunov's second method for continuous ordinary differential equations. The stability of solutions to DEPCAG (2) is considered in Akhmet et. al. [6] by the use of Lyapunov functions for systems that are related to the system in Eq. (2) through a perturbation relation, based on Malkin's [25] approach to stability problem. However, to the best of our knowledge, absolutely continuous solutions to DEPCAG (2) have not been considered in literature so far.

The novelty of the paper is that the existence of absolutely continuous solutions is established in a constructive way, and the stability of the solutions is obtained without f having to be continuous in the first argument. Rather, we use nonsmooth Lyapunov functions to a system given by Eq. (2). We also point out that the qualitative study carried out here for DEPCAG is fairly new as compared to the method based on reducing DEPCAG to discrete equations presented in [3, 8, 18]. More specifically, this paper establishes the Lyapunov's second method for DEPCAG's, so that for the stability, we make use of Lyapunov functions for the system given in Eq. (2). Utilizing one of the results established in this work, we examine the stability of a specific class of DEPCAGs, particularly focusing on the logistic equation with a piecewise constant argument of generalized type, as discussed by Aruğaslan and Güzel [8].

The organization of the present paper is as follows: After a thorough introduction to the topic, in Sect. 2 we provide essential preliminaries required to understand the paper. In Sect. 3, we establish the existence of AC solutions to DEPCAG (2). In Sect. 4, we investigate the stability analysis of DEPCAG (2) based on Lyapunov's second method. As an application of the theoretical framework developed in the preceding sections, Sect. 5 is devoted to

analyzing the stability of an epidemic model and a logistic growth model characterized by a piecewise constant argument of generalized type. The results are summarized in Sect. 6.

2 Preliminaries

In this section, we provide necessary preliminaries that will be used throughout this paper. The notation $\mathbb{N}_0$ denotes the set of nonnegative integers. By $\|\cdot\|$, we mean the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Below, we give the definition of an absolutely continuous function, which is taken from Gordon [19].

Definition 1 (See [19, Definition 4.1 (b)]) A function $x : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ provided given any $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$\sum_{i=1}^n |x(d_i) - x(c_i)| < \varepsilon$$

whenever a non-overlapping collection $\{[c_i, d_i] : 1 \leq i \leq n\}$ of intervals in $[a, b]$ satisfy $\sum_{i=1}^n (d_i - c_i) < \delta$.

An absolutely continuous (AC) function $x(\cdot)$ is considered a solution to Eq. (2) on $\mathbb{R}^+$ if it satisfies (2) almost everywhere (a.e.) on the intervals $t \in [\theta_i, \theta_{i+1})$, for each $i \in \mathbb{N}_0$, with respect to the Lebesgue measure. From now on, a solution to the Eq. (2) will be understood as an AC function.

Let $\mathcal{B}^n$ denote the Borel σ -algebra of $\mathbb{R}^n$ and $\mathcal{L}$ the Lebesgue σ -algebra of $\mathbb{R}$. Denote by $\mathcal{L} \times \mathcal{B}^n$ the product σ -algebra between $\mathcal{L}$ and $\mathcal{B}^n$. For each $i \in \mathbb{N}_0$, let $y_i \in \mathbb{R}^n$ be an arbitrarily fixed vector, and define the function $F_i : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $F_i(t, x) = f(t, x, y_i)$, where f is given in (2). Given that F_i may be discontinuous, the corresponding differential equation

$$\dot{x}(t) = F_i(t, x(t))$$

is a discontinuous. The results obtained in the present work concerning Eq. (2) are established from discontinuous differential equations. A comprehensive approach to discontinuous differential equations can be found in [10, 20, 30]. Our results concerning the existence of solutions to Eq. (2) are mainly based on the following result.

Theorem 1 (See [33, Corollary 2.4.5]) Let $g : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be such that

- (i) for each $x \in \mathbb{R}^n$, $g(\cdot, x)$ is Lebesgue measurable;
- (ii) there is a Lebesgue integrable function $k : [a, b] \rightarrow [0, \infty)$ such that

$$\|g(t, x_1) - g(t, x_2)\| \leq k(t)\|x_1 - x_2\|$$

for all $x_1, x_2 \in \mathbb{R}^n$, and a.e. $t \in [a, b]$. Then, for any $\xi \in \mathbb{R}^n$, the initial value problem

$$\begin{aligned}\dot{x}(t) &= g(t, x(t)) \quad \text{a.e. } t \in [a, b], \\ x(a) &= \xi,\end{aligned}$$

admits a unique solution.

3 Existence of solutions

Given a closed and bounded interval I in $\mathbb{R}$ and a function $h : I \rightarrow \mathbb{R}^n$, we will denote by $\|h\|_0$ the supremum norm of h . We denote the set of all continuous functions from I to $\mathbb{R}^n$ by $C(I, \mathbb{R}^n)$. When equipped with the supremum norm $\|\cdot\|_0$, this is a Banach space. The existence of solutions to Eq. (2) have been established in Theorems 2 and 3 based on the following hypotheses:

- (H₁) The function $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is $\mathcal{L} \times \mathcal{B}^n$ -measurable in the third argument.
- (H₂) For each $t \in [0, \infty)$, the function $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is such that $f(t, 0, 0) = 0$.
- (H₃) There exists a constant $\theta > 0$ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{N}_0$.
- (H₄) There exists a constant $l > 0$ satisfying

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq l(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, and a.e. $t \in [0, \infty)$;

- (H₅) $2l\theta < 1$.

Theorem 2 Assume that $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the hypotheses (H₁)–(H₅). Further, suppose there exist $\gamma > 0$ and an integrable function $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\|f(t, x, y)\| \leq \gamma(\|x\| + \|y\|) + c(t) \quad (3)$$

for any $x, y \in \mathbb{R}^n$ and any $t \in \mathbb{R}^+$. Then, for every $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, we have the following:

- (i) If $t_0 = \theta_j$ for some $j \in \mathbb{N}_0$, then there exists a unique solution $x(\cdot)$ to Eq. (2) on $[t_0, \infty)$ with $x(t_0) = x_0$;
- (ii) If $t_0 \in (\theta_j, \theta_{j+1}]$ for some $j \in \mathbb{N}_0$, then there exists a solution $y(\cdot)$ to Eq. (2) on $[\theta_j, \theta_{j+1}]$ with $y(t_0) = x_0$.

Proof Let $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ be arbitrarily fixed. Suppose that $t_0 = \theta_j$ for some $j \in \mathbb{N}_0$. From Theorem 1, solution $y_j(\cdot)$ to the Eq. (2) on $[\theta_j, \theta_{j+1}]$ such that $y_j(t_0) = x_0$ exists and is unique. If $y_j(\theta_{j+1}) = w_1$, then from Theorem 1, a unique solution $y_{j+1}(\cdot)$ to Eq. (2) on $[\theta_{j+1}, \theta_{j+2}]$ such that $y_{j+1}(\theta_{j+1}) = w_1$ exists. Employing Theorem 1 repeatedly together with the principle of mathematical induction, we obtain a sequence $\{y_{j+i}\}_{i \in \mathbb{N}_0}$ of solutions $y_{j+i} : [\theta_{j+i}, \theta_{j+i+1}] \rightarrow \mathbb{R}^n$ to Eq. (2) on $[\theta_{j+i}, \theta_{j+i+1}]$ such that $y_{j+i}(\theta_{j+i}) = w_i$ and $w_0 = x_0$. Thus, the function $x : [t_0, \infty) \rightarrow \mathbb{R}^n$ defined by $x(t) = y_{j+i}(t)$, for every $t \in [\theta_{j+i}, \theta_{j+i+1})$ and for each $i \in \mathbb{N}_0$, is the unique solution to Eq. (2) on $[t_0, \infty)$ with $x(t_0) = x_0$.

Now, suppose that $t_0 \in (\theta_j, \theta_{j+1}]$ for some $j \in \mathbb{N}_0$, and $\{y_k\}_{k \in \mathbb{N}_0}$ is a sequence of functions $y_k : [\theta_j, \theta_{j+1}] \rightarrow \mathbb{R}^n$ obeying $y_0(t) = x_0$ and

$$y_{k+1}(t) = x_0 + \int_{t_0}^t f(s, y_k(s), y_k(\theta_j)) ds \quad (4)$$

for all $k \geq 0$ and all $t \in [\theta_j, \theta_{j+1}]$. The hypothesis (H_1) together with the condition 3 guarantees the existence of the integral on right side of (4). Then, for each $t \in [\theta_j, \theta_{j+1}]$, we have

$$\begin{aligned} \|y_1(t) - y_0(t)\| &= \|y_1(t) - x_0\| \\ &\leq \int_{\theta_j}^{\theta_{j+1}} \|f(s, y_0(s), y_0(\theta_j))\| ds \\ &\stackrel{(H_2)}{=} \int_{\theta_j}^{\theta_{j+1}} \|f(s, y_0(s), y_0(\theta_j)) - f(s, 0, 0)\| ds \\ &\stackrel{(H_4)}{\leq} \int_{\theta_j}^{\theta_{j+1}} l(\|y_0(s)\| + \|y_0(\theta_j)\|) ds \\ &\stackrel{(H_3)}{\leq} 2l\theta \|x_0\|. \end{aligned}$$

If $k \in \mathbb{N}_0 \setminus \{0\}$, then for each $t \in [\theta_j, \theta_{j+1}]$, we have

$$\begin{aligned} \|y_{k+1}(t) - y_k(t)\| &= \left\| \int_{t_0}^t (f(s, y_k(s), y_k(\theta_j)) - f(s, y_{k-1}(s), y_{k-1}(\theta_j))) ds \right\| \\ &\leq \int_{\theta_j}^{\theta_{j+1}} \|f(s, y_k(s), y_k(\theta_j)) - f(s, y_{k-1}(s), y_{k-1}(\theta_j))\| ds \\ &\stackrel{(H_4)}{\leq} \int_{\theta_j}^{\theta_{j+1}} l(\|y_k(s) - y_{k-1}(s)\| + \|y_k(\theta_j) - y_{k-1}(\theta_j)\|) ds \\ &\stackrel{(H_3)}{\leq} 2l\theta \|y_k - y_{k-1}\|_0 \\ &\quad \vdots \\ &\leq (2l\theta)^k \|y_1 - y_0\|_0 \\ &\leq (2l\theta)^{k+1} \|x_0\|, \end{aligned}$$

which yields

$$\|y_{k+1} - y_k\|_0 \leq (2l\theta)^{k+1} \|x_0\|.$$

Keeping in mind (H_5) , we conclude that the sequence $\{y_k\}_{k \in \mathbb{N}_0}$ is Cauchy in $C([\theta_j, \theta_{j+1}], \mathbb{R}^n)$. Hence there is $y \in C([\theta_j, \theta_{j+1}], \mathbb{R}^n)$ for which $\{y_k\}_{k \in \mathbb{N}_0}$ converges uniformly to y on $[\theta_j, \theta_{j+1}]$. Letting $k \rightarrow \infty$ in Eq. (4), we get

$$y(t) = x_0 + \int_{t_0}^t f(s, y(s), y(\theta_j)) ds.$$

Thus,

$$\dot{y}(t) = f(t, y(t), y(\theta_j)) \quad \text{a.e. } t \in [\theta_j, \theta_{j+1}],$$

and then $y(\cdot)$ is a solution to Eq. (2) on $[\theta_j, \theta_{j+1}]$ with $y(t_0) = x_0$. $\square$

Remark 1 We see from Theorem 2 (ii) that the solution need not be unique if $t_0 \in (\theta_j, \theta_{j+1}]$. We present a counterexample to illustrate that the initial value problem corresponding to the DEPCAG may fail to have a unique solution when the initial time lies strictly between two consecutive partition points, even if all the hypotheses (H₁)–(H₅) are satisfied. We illustrate this in the following example.

Example 1 Let $\{\theta_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+$ be a strictly increasing sequence such that $\theta_{j+1} - \theta_j = \theta > 0$ for all $j \in \mathbb{N}$. Define the piecewise constant deviating argument $\beta(t) = \theta_j$ for $t \in [\theta_j, \theta_{j+1})$. Consider the differential equation with piecewise constant argument of generalized type

$$\dot{x}(t) = f(t, x(t), x(\beta(t))), \quad x(t_0) = 0, \quad (5)$$

where $t_0 \in (\theta_j, \theta_{j+1})$ for some j , and $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(t, x, y) = \begin{cases} \sqrt{|y|} & \text{if } |y| \geq \delta, \\ \psi(y) & \text{if } |y| < \delta, \end{cases}$$

where $\delta > 0$ is arbitrarily fixed and $\psi : [-\delta, \delta] \rightarrow \mathbb{R}$ is a continuous function given by

$$\psi(y) = \sqrt{\delta} \cdot \sin\left(\frac{\pi|y|}{2\delta}\right).$$

We see that $f(t, 0, 0) = \psi(0) = 0$ for all t . Thus (H₂) holds. Now, taking $\theta = 0.2$ we see that $\theta_{j+1} - \theta_j = \theta$ for all j , that is, (H₃) is true. Note that

$$\lim_{y \rightarrow \delta^-} \psi(y) = f(t, x, \delta) = \sqrt{\delta} \quad \text{and} \quad \lim_{y \rightarrow -\delta^+} \psi(y) = f(t, x, -\delta) = \sqrt{\delta}.$$

Also, since $\psi'(y) = \frac{\pi}{2\sqrt{\delta}} \cos\left(\frac{\pi y}{2\delta}\right)$ for $y \in (0, \delta)$, we have $\psi'_+(0) = \frac{\pi}{2\sqrt{\delta}}$. Next, for $y_1, y_2 \in [-\delta, \delta]$, we have $|y_1|, |y_2| \in [0, \delta]$. Therefore, in the view of mean value theorem, there exists y_3 between $|y_1|$ and $|y_2|$ satisfying

$$|\psi(y_1) - \psi(y_2)| = |\psi'(y_3)(|y_1| - |y_2|)|.$$

On the other hand, for $|y_1|, |y_2| \geq \delta$, again in the view of mean value theorem, there exists y_3 between $|y_1|$ and $|y_2|$ satisfying

$$\left| \sqrt{|y_1|} - \sqrt{|y_2|} \right| = \frac{1}{2\sqrt{y_3}} ||y_1| - |y_2||.$$

Further, for $|y_1| \geq \delta$ and $|y_2| \leq \delta$, note that

$$\begin{aligned} |f(t, x_1, y_1) - f(t, x_2, y_2)| &= |f(t, x_1, y_1) - f(t, x, \delta) + f(t, x, \delta) - f(t, x_2, y_2)| \\ &\leq |f(t, x_1, y_1) - f(t, x, \delta)| + |f(t, x, \delta) - f(t, x_2, y_2)| \\ &= \left| \sqrt{|y_1|} - \sqrt{\delta} \right| + \left| \sqrt{\delta} - \sqrt{\delta} \cdot \sin \left(\frac{\pi |y_2|}{2\delta} \right) \right| \\ &\leq \frac{1}{2\sqrt{\delta}} ||y_1| - \delta| + \frac{\pi}{2\sqrt{\delta}} |\delta - |y_2|| \\ &\leq \frac{\pi}{2\sqrt{\delta}} ||y_1| - \delta| + \frac{\pi}{2\sqrt{\delta}} |\delta - |y_2|| \\ &= \frac{\pi}{2\sqrt{\delta}} (|y_1| - |y_2|) \\ &\leq \frac{\pi}{2\sqrt{\delta}} |y_1 - y_2|. \end{aligned}$$

Thus, the constant $l > 0$ in (H_4) is given by $l = \frac{\pi}{2\sqrt{\delta}}$, and subsequently, (H_1) is also valid, since f , as we seen previously, is continuous (in fact, f is Lipschitz continuous with Lipschitz constant $l = \frac{\pi}{2\sqrt{\delta}}$). Next, by taking $\delta = 1$ and $\theta = 0.2$, we find

$$2l\theta = 2 \cdot \frac{\pi}{2} \cdot 0.2 = \pi \cdot 0.2 \approx 0.628 < 1,$$

which implies that (H_5) is satisfied. Thus, all the hypotheses (H_1) – (H_5) hold. However, we shall now demonstrate that the considered DEPCAG (5) has no unique solution.

Let $t_0 \in (\theta_j, \theta_{j+1})$. Note that since $x(\theta_j)$ is not specified by the initial condition $x(t_0) = 0$, multiple consistent values of $x(\theta_j)$ may be assumed depending on different prior extensions. We now present two different functions that both satisfy the equation and the initial condition at t_0 given in (5), but differ due to different values at $x(\theta_j)$. For our first solution, define $x_1(t) = 0$ for all $t \in \mathbb{R}^+$. Then

$$x_1(\theta_j) = 0, \quad \dot{x}_1(t) = f(t, 0, 0) = 0 \quad \text{on} \quad (\theta_j, \theta_{j+1}).$$

Hence, x_1 satisfies the equation and the initial condition $x_1(t_0) = 0$ given in (5). Now, for the second solution, fix $y_0 > 0$, and define

$$x_2(t) = \sqrt{y_0}(t - t_0) \quad \text{for} \quad t \in [t_0, \theta_{j+1}).$$

We extend x_2 backwards to $t = \theta_j$ by setting $x_2(\theta_j) = -y_0$. Then

$$\dot{x}_2(t) = \sqrt{y_0} = f(t, x_2(t), x_2(\theta_j)) = f(t, x_2(t), y_0).$$

Thus, x_2 satisfies the equation in (5) on (θ_j, θ_{j+1}) . Also, $x_2(t_0) = \sqrt{y_0}(t_0 - t_0) = 0$, so it also satisfies the initial condition given in (5). Hence x_1 and x_2 defined above are solutions

of DEPCAG (5). Although both x_1 and x_2 satisfy the same initial value at t_0 , they differ in the value of $x(\theta_j)$ due to different possible extensions prior to t_0 . Since the equation in DEPCAG (5) depends on $x(\theta_j)$, the same initial value at t_0 leads to different right-hand sides on (θ_j, θ_{j+1}) . Therefore, both x_1 and x_2 defined above are valid solutions of DEPCAG (5), yet they are not identical. Hence, DEPCAG (5) has no unique solution. This confirms the assertion of Theorem 2 (ii) that uniqueness may not hold in this case.

Theorem 3 Assume that $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the hypotheses (H_1) – (H_5) . Further, suppose there exist $\gamma > 0$ and an integrable function $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\|f(t, x, y)\| \leq \gamma (\|x\| + \|y\|) + c(t)$$

for any $x, y \in \mathbb{R}^n$ and any $t \in \mathbb{R}^+$. Then, for every $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, a solution $x(\cdot)$ to Eq. (2) on $[0, \infty)$ with $x(t_0) = x_0$ exists.

Proof Let $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ be arbitrary. For $t_0 = 0$, by Theorem 2 (i), a unique solution $x(\cdot)$ to Eq. (2) on $[0, \infty)$ such that $x(t_0) = x_0$ exists. If $t_0 \in (\theta_j, \theta_{j+1}]$ for some $j \in \mathbb{N}_0$, then from Theorem 2 (ii) there exists a solution $y(\cdot)$ to Eq. (2) on $[\theta_j, \theta_{j+1}]$ such that $y(t_0) = x_0$. Thus, by the virtue of Theorem 2, we obtain a unique solution $z(\cdot)$ to Eq. (2) on $[\theta_{j+1}, \infty)$ such that $z(\theta_{j+1}) = y(\theta_{j+1})$. Now, for $j = 0$, the function $x : [0, \infty) \rightarrow \mathbb{R}^n$ defined by

$$x(t) = \begin{cases} y(t) & \text{for } t \in [\theta_0, \theta_1], \\ z(t) & \text{for } t \in [\theta_1, \infty), \end{cases}$$

is a solution to Eq. (2) on $[0, \infty)$ for which $x(t_0) = x_0$. On the other hand, for $j \in \mathbb{N}_0 \setminus \{0\}$, we apply Theorem 2 to obtain a solution $y_j(\cdot)$ to Eq. (2) on $[\theta_{j-1}, \theta_j]$ with $y_j(\theta_j) = y(\theta_j)$. Thus, we can apply Theorem 2 j times to obtain the solutions $y_i(\cdot)$ to Eq. (2) on $[\theta_{i-1}, \theta_i]$ such that $y_i(\theta_i) = y_{i+1}(\theta_i)$, $i \in \{1, \dots, j\}$, and $y_{j+1}(\theta_j) = y(\theta_j)$. Thus, the function $x : [0, \infty) \rightarrow \mathbb{R}^n$ given by

$$x(t) = \begin{cases} y_i(t) & \text{for } t \in [\theta_{i-1}, \theta_i], i \in \{1, \dots, j\}, \\ y(t) & \text{for } t \in [\theta_j, \theta_{j+1}], \\ z(t) & \text{for } t \in [\theta_{j+1}, \infty), \end{cases}$$

is a solution to Eq. (2) on $[0, \infty)$ such that $x(t_0) = x_0$. □

4 Stability of solutions

In this section, we develop Lyapunov's second method and apply it to examine the stability characteristics of the DEPCAG system described by (2). The notion of Lyapunov stability for the system in Eq. (2) can be formulated similarly to the approaches found in [26, 31, 34]. We begin by introducing definitions related to different types of stability. Let $S(x_0, t_0)$ represent the set of all solutions $x(\cdot)$ of Eq. (2) defined on $[0, \infty)$ with the initial condition $x(t_0) = x_0$. When $x \in S(x_0, t_0)$, we also use the notation $x(t, t_0, x_0)$ to refer to the function $x(t)$.

Definition 2 The equilibrium point $x = 0$ of Eq. (2) is said to be stable provided for every $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \in [t_0, \infty)$ whenever $\|x_0\| < \delta$.

If δ can be chosen independently of t_0 , that is, $\delta = \delta(\varepsilon)$, then the equilibrium $x = 0$ of Eq. (2) is said to be *uniformly stable*.

Definition 3 The equilibrium $x = 0$ of Eq. (2) is said to be uniformly asymptotically stable provided it is uniformly stable, and there exists a constant $\delta_0 > 0$ such that, for every $\varepsilon > 0$ and any $t_0 \geq 0$, there exists $T(\varepsilon) > 0$ independent of t_0 , for which $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon)$ whenever $\|x_0\| < \delta_0$.

We can also define the concepts of stability (in the sense of Lyapunov) for the solution $x \equiv 0$ of Eq. (2) with the initial condition $x(0) = x_0$. These definitions are presented below.

Definition 4 The solution $x \equiv 0$ of Eq. (2) with the initial condition $x(0) = x_0$ is said to be stable provided for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x(t, 0, x_0)\| < \varepsilon$ for all $t \geq 0$ whenever $\|x_0\| < \delta$.

Definition 5 The solution $x \equiv 0$ of Eq. (2) with the initial condition $x(0) = x_0$ is said to be asymptotically stable provided it is stable and there exists $\delta > 0$ such that $\lim_{t \rightarrow \infty} \|x(t, 0, x_0)\| = 0$ whenever $\|x_0\| < \delta$.

The results concerning Lyapunov's second method to DEPCAG can be obtained just as obtained for ordinary differential equations. For completeness and ease of reading, we enunciate and prove results regarding various stabilities. These results are separately proved in Theorems 4, 5, 6, 7, and 8, respectively.

Definition 6 We define class $\mathcal{K}$ to be the set of continuous function $\psi : [0, r_1] \rightarrow \mathbb{R}^+$ (respectively, $\psi : [0, \infty) \rightarrow \mathbb{R}^+$) such that $\psi(0) = 0$ and ψ is strictly increasing on $[0, r_1]$ (respectively, on $[0, \infty)$).

In what follows, $B(h)$ denotes the open ball of radius h centered at the origin,

$$B(h) = \{x \in \mathbb{R}^n : \|x\| < h\}.$$

Below we list conditions on $V : \mathbb{R}^+ \times B(h) \rightarrow \mathbb{R}$, which are required to prove the results concerning stability.

(C₁) For each $t \geq 0$, $V(t, 0) = 0$.

(C₂) For each $t \geq 0$, there exists $r > 0$ such that for each $x \in B(r)$, $V(t, x) \geq \psi(\|x\|)$, where $\psi \in \mathcal{K}$.

(C₃) $\frac{d}{dt} V(t, \phi(t)) \leq 0$ for a.e. $t \geq 0$ and for all $\phi \in S(x_0, t_0)$ with $\|\phi(t)\| < h_1$, $h_1 < h$.

(C₄) For some $r > 0$, $\psi_1(\|x\|) \leq V(t, x) \leq \psi_2(\|x\|)$ for all $t \geq 0$ and all $x \in B(r)$, where $\psi_1, \psi_2 \in \mathcal{K}$.

(C₅) There is $\psi_3 \in \mathcal{K}$ such that $\frac{d}{dt}V(t, \phi(t)) \leq -\psi_3(\|\phi(t)\|)$ for a.e. $t \geq 0$ and for all $\phi \in S(x_0, t_0)$ with $\|\phi(t)\| < r$.

Theorem 4 Suppose that a locally Lipschitz continuous function $V : \mathbb{R}^+ \times B(h) \rightarrow \mathbb{R}$ exists such that (C₁)–(C₃) hold. Then the equilibrium $x = 0$ of Eq. (2) is stable.

Proof If necessary, we can redefine h_1 so that $V(t, x) \geq \psi(\|x\|)$ for $\|x\| < h_1$ and $t \geq 0$. Take arbitrary $\varepsilon > 0$ such that $\varepsilon < h_1$ and $t_0 \geq 0$. Since V is continuous and $V(t_0, 0) = 0$, we can find $\delta > 0$ such that $V(t_0, x_0) < \psi(\varepsilon)$ for $\|x_0\| \leq \delta$. Without any restrictions, we can assume that $\delta < \varepsilon$. Let $\phi \in S(x_0, t_0)$ be such that $\|\phi(t)\| < h_1$ for each $t \geq 0$. Then, in the view of condition (C₃), we have

$$\begin{aligned} V(t_2, \phi(t_2)) &= V(t_1, \phi(t_1)) + \int_{t_1}^{t_2} \frac{d}{ds} V(s, \phi(s)) ds \\ &\leq V(t_1, \phi(t_1)) \end{aligned}$$

for each $t_1, t_2 \geq 0$ with $t_2 \geq t_1$. This yields that $V(t, \phi(t))$ is nonincreasing in t . Thus,

$$V(t, \phi(t)) \leq V(t_0, \phi(t_0)) = V(t_0, x_0) < \psi(\varepsilon)$$

for each $t \geq t_0$. Now, for some $t > t_0$, if $\|\phi(t)\|$ reaches the value ε , then we get a contradiction as

$$\psi(\varepsilon) > V(t, \phi(t)) \geq \psi(\|\phi(t)\|) = \psi(\varepsilon).$$

Thus, $\|\phi(t)\| < \varepsilon$ for all $t \geq t_0$ and the equilibrium $x = 0$ is stable. $\square$

Remark 2 The function V defined in the Theorem 4 is a Lyapunov function.

Theorem 5 Suppose that there exists a locally Lipschitz continuous function $V : \mathbb{R}^+ \times B(h) \rightarrow \mathbb{R}$ such that (C₁), (C₃), and (C₄) hold. Then the equilibrium $x = 0$ is uniformly stable.

Proof Without any restrictions, we can assume that $\psi_1(\|x\|) \leq V(t, x) \leq \psi_2(\|x\|)$ for $\|x\| < h_1$ and $t \geq 0$. Select δ and ε such that $0 < \delta < \varepsilon < h_1$ and $\psi_2(\delta) < \psi_1(\varepsilon)$. Then we have

$$V(t_0, x_0) \leq \psi_2(\|x_0\|) < \psi_2(\delta) < \psi_1(\varepsilon).$$

for $t_0 \geq 0$ and $\|x_0\| < \delta$. Next, consider $\phi \in S(x_0, t_0)$ such that $\|\phi(t)\| < h_1$ for each $t \geq 0$. Since the function $V(t, \phi(t))$ is nonincreasing in t , we get

$$V(t, \phi(t)) < \psi_1(\varepsilon)$$

for each $t \geq t_0$. Therefore, $\psi_1(\|\phi(t)\|) < \psi_1(\varepsilon)$ for each $t \geq t_0$. Also, since ψ_1 is an invertible function, we may conclude that $\|\phi(t)\| < \varepsilon$ for each $t \geq t_0$. Thus, the equilibrium $x = 0$ is uniformly stable. $\square$

Theorem 6 Suppose that a locally Lipschitz continuous function $V : \mathbb{R}^+ \times B(h) \rightarrow \mathbb{R}$ exists such that (C_1) , (C_4) , and (C_5) hold. Then the equilibrium $x = 0$ of Eq. (2) is uniformly asymptotically stable.

Proof Without any restrictions, we can assume that $0 < r < h$. From Theorem 5, the equilibrium $x = 0$ is uniformly stable. Now, select δ_1, δ_2 with $0 < \delta_2 < \delta_1$ such that $\psi_2(\delta_1) < \psi_1(r)$ and $\psi_2(\delta_2) < \psi_1(\varepsilon)$, where $\varepsilon > 0$ is arbitrarily fixed such that $\varepsilon \leq r$. Define

$$T = \frac{\psi_1(r)}{\psi_3(\delta_2)},$$

and fix $t_0 \geq 0$ and $x_0 \in B(\delta_1)$. Let $\phi \in S(x_0, t_0)$ be such that $\|\phi(t)\| < r$ for all $t \geq 0$. We claim that $\|\phi(t^*)\| < \delta_2$ for some $t^* \in [t_0, t_0 + T]$. Otherwise, if $\|\phi(t)\| \geq \delta_2$ for each $t \in [t_0, t_0 + T]$, then, for each $t \in [t_0, t_0 + T]$, we obtain

$$\begin{aligned} 0 < \psi_1(\delta_2) &\leq \psi_1(\|\phi(t)\|) \leq V(t, \phi(t)) \\ &= V(t_0, \phi(t_0)) + \int_{t_0}^t \frac{d}{ds} V(s, \phi(s)) ds \\ &\leq \psi_2(\|x_0\|) - \int_{t_0}^t \psi_3(\|\phi(s)\|) ds \\ &\leq \psi_2(\delta_1) - \int_{t_0}^t \psi_3(\delta_2) ds. \end{aligned}$$

In particular, if $t = t_0 + T$, then $0 < \psi_2(\delta_1) - T\psi_3(\delta_2) = \psi_2(\delta_1) - \psi_1(r) < 0$ and we get a contradiction. Thus, such t^* exists. Next, we see that if $t \geq t^*$, then

$$\begin{aligned} \psi_1(\|\phi(t)\|) &\leq V(t, \phi(t)) \leq V(t^*, \phi(t^*)) \\ &\leq \psi_2(\|\phi(t^*)\|) \\ &\leq \psi_2(\delta_2) \\ &< \psi_1(\varepsilon). \end{aligned}$$

Now, since $\psi_1 \in \mathcal{K}$, $\|\phi(t)\| < \varepsilon$ for each $t \geq t^*$ and then for each $t \geq t_0 + T$. Thus, the equilibrium $x = 0$ is uniformly asymptotically stable. $\square$

As can be seen below, we can also state stability results for the solution $x \equiv 0$ of Eq. (2) with $x(0) = x_0$.

Theorem 7 Suppose that a locally Lipschitz continuous function $V : \mathbb{R}^+ \times B(h) \rightarrow \mathbb{R}$ exists such that (C_1) , (C_2) hold and (C_3) holds with $t_0 = 0$. Then the solution $x \equiv 0$ is stable.

Proof The proof can be produced along the same lines as Theorem 4. $\square$

Theorem 8 *Suppose that a locally Lipschitz continuous function $V : \mathbb{R}^+ \times B(h) \rightarrow \mathbb{R}$ exists such that (C_1) , (C_4) hold and (C_5) holds with $t_0 = 0$. Then the solution $x \equiv 0$ is asymptotically stable.*

Proof The proof can be produced along the same lines as that of Theorem 6. $\square$

5 Applications

To illustrate the applications of the theoretical results established in the preceding sections, we now investigate the stability of specific classes of DEPCAG systems describing an epidemic model and a logistic model. The epidemic model is useful in studying various biological processes while the logistic model is widely used in studying the population dynamics of a single species in an environment with limited resources. We emphasize that the difference between the methodology used here and that used in Akhmet et al. [6] for the stability study is that we use Lyapunov's second method for DEPCAGs, which means that we use Lyapunov functions for the system given in Eq. (2).

5.1 Epidemic model

Motivated by [11], in this subsection, we study the stability of the system described by the nonlinear epidemic model

$$\dot{x}(t) = f_2(1 - x(t), x(\beta(t))) - cx(t), \quad (6)$$

where $c > 0$ is a given parameter, $x(t) \in \mathbb{R}$ and $t \in [0, \infty)$. The existence of solutions is guaranteed by Theorem 3. Suppose the function f_2 in Eq. (6) is given by

$$f_2(u, v) = c(u - 1)(\sin^2(v) - 1) \quad (7)$$

for every $u, v \in \mathbb{R}$. Then, the equilibrium $x = 0$ of Eq. (6) is uniformly stable. Indeed, assume $x(\cdot)$ a solution of Eq. (6) on $[0, \infty)$. Then,

$$\begin{aligned} \dot{x}(t) &= f_2(1 - x(t), x(\beta(t))) - cx(t) \\ &= c(1 - x(t) - 1)(\sin^2(x(\beta(t))) - 1) - cx(t) \\ &= -cx(t) \sin^2(x(\beta(t))) \end{aligned}$$

for a.e. $t \in [0, \infty)$. Now, we may take the Lyapunov function $V(t, x) = x^2$ to conclude that

$$\begin{aligned} \frac{d}{dt} V(t, x(t)) &= 2x(t)\dot{x}(t) \\ &= -2cx^2(t) \sin^2(x(\beta(t))) \\ &\leq 0 \end{aligned}$$

for a.e. $t \in [0, \infty)$. Hence, it is a consequence of Theorem 5 that the equilibrium $x = 0$ of Eq. (6) is uniformly stable.

5.2 Logistic model

In this subsection, we focus on a logistic growth model incorporating a generalized type of piecewise constant argument defined as follows

$$\dot{x}(t) = (a - bx(\beta(t)))x(t), \quad (8)$$

where a and b are both to the left or right of zero. The stability investigation of the fixed point $x = 0$ is carried out, by Aruğaslan and Güzel [8], after reducing Eq. (8) into a non-autonomous difference equation. If $b < 0$, then keeping in mind Theorem 8, we can also conclude that the solution $x \equiv 0$ is asymptotically stable. Indeed, to see that, we can use the Lyapunov function $V(t, x) = x^2$. Let $x(\cdot)$ be a solution of Eq. (8) satisfying $x(0) = x_0$. If $x_0 = 0$ and $t \in [\theta_0, \theta_1)$, then Eq. (8) reduces to

$$\dot{x}(t) = ax(t),$$

which can be solved to give $x(t) = x_0 e^{at} = 0$ for $t \in [\theta_0, \theta_1)$. Now, if $x(\theta_1) = x_1 = 0$ and $t \in [\theta_1, \theta_2)$, then Eq. (8) reduces to

$$\dot{x}(t) = ax(t),$$

and whose solution is given by $x(t) = x_1 e^{a(t-\theta_1)} = 0$ for $t \in [\theta_1, \theta_2)$. By the principle of induction, it follows that $x(t) = 0$ for each $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$. Now, take $h > 0$ such that $h < \frac{a}{b}$. For $0 < x_0 < h$, if $x(t) > 0$ for $t \in (0, c_1)$ with $0 < c_1 \leq \theta_1$, then Eq. (8) reduces to

$$\dot{x}(t) = (a - bx_0)x(t)$$

for $t \in [\theta_0, c_1)$. Thus, we have its solution as

$$x(t) = x_0 e^{(a-bx_0)t},$$

and then $c_1 = \theta_1$ and $0 < x(t) \leq x_0$ for each $t \in [\theta_0, \theta_1]$. Next, suppose that $x(\theta_1) = x_1$ and $x(t) > 0$ for $t \in (\theta_1, c_2)$, with $\theta_1 < c_2 \leq \theta_2$. Then Eq. (8) reduces to

$$\dot{x}(t) = (a - bx_1)x(t)$$

for $t \in [\theta_1, c_2)$, whose solution is

$$x(t) = x_1 e^{(a-bx_1)(t-\theta_1)},$$

and then $c_2 = \theta_2$ and $0 < x(t) \leq x_0$ for each $t \in [\theta_1, \theta_2]$. The principle of induction allows us to conclude that $0 < x(t) \leq x_0$ for each $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$. If $-h < x_0 < 0$, then take $c_1 \in (0, \theta_1]$ such that $x(t) < 0$ for $t \in (0, c_1)$. Therefore, Eq. (8) becomes

$$\dot{x}(t) = (a - bx_0)x(t)$$

for $t \in [\theta_0, c_1)$, whose solution is

$$x(t) = x_0 e^{(a-bx_0)t},$$

and then $c_1 = \theta_1$ and $x_0 \leq x(t) < 0$ for each $t \in [\theta_0, \theta_1]$. If $x(\theta_1) = x_1$, then take $c_2 \in (\theta_1, \theta_2]$ such that $x(t) < 0$ for $t \in (\theta_1, c_2)$. Therefore, Eq. (8) becomes

$$\dot{x}(t) = (a - bx_1)x(t)$$

for $t \in [\theta_1, c_2)$, whose solution is

$$x(t) = x_1 e^{(a-bx_1)(t-\theta_1)},$$

and then $c_2 = \theta_2$ and $x_0 \leq x(t) < 0$ for each $t \in [\theta_1, \theta_2]$. The principle of induction allows us to infer that $x_0 \leq x(t) < 0$ for each $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$. Therefore, if $\|x_0\| < h$, then the solution $x(\cdot)$ of Eq. (8) (with $x(0) = x_0$) is such that $\|x(t)\| < h$ for all $t \geq 0$. Moreover,

$$\begin{aligned} \frac{d}{dt} V(t, x(t)) &= 2x(t)\dot{x}(t) \\ &= 2(a - bx(\beta(t)))(x(t))^2 \\ &\leq 2(a - bh)(x(t))^2 \\ &= -\psi_3(\|x(t)\|) \end{aligned}$$

for $t \in (\theta_i, \theta_{i+1})$, $i \in \mathbb{N}_0$, where $\psi_3(u) = 2(bh - a)u^2$. By virtue of Theorem 8, the solution $x \equiv 0$ of Eq. (8) is asymptotically stable.

6 Conclusion

This paper has investigated the existence and stability properties of absolutely continuous (AC) solutions for a class of differential equations characterized by a generalized form of piecewise constant argument. We have established sufficient conditions for the existence of absolutely continuous (AC) solutions to the DEPCAG system given by Eq. (2). Subsequently, we examined the stability of these solutions using Lyapunov's second method, which was adapted to accommodate nonsmooth Lyapunov functions. To illustrate the applicability of our theoretical findings, we analyzed the stability of the trivial solution $x \equiv 0$ in the context of epidemic model as well as logistic model with generalized piecewise constant argument. Our analysis highlights the significant impact of the generalized piecewise constant argument on the qualitative behavior of the solutions.

Looking ahead, the present study can be extended in several directions. For instance, one may investigate other qualitative properties such as the existence of bounded or almost periodic solutions to DEPCAG (2), as well as explore the influence of initial data on solution behavior. Furthermore, the study of alternative solution concepts, such as Filippov and Carathéodory solutions, may offer valuable insights and constitute promising avenues for future research.

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References

1. Abildayeva, A.D., Assanova, A.T., Imanchiev, A.E.: A multi-point problem for a system of differential equations with piecewise-constant argument of generalized type as a neural network model. *Eurasian Math. J.* **13**(2), 8–17 (2022)
2. Akhmet, M.U.: Almost periodic solutions of differential equations with piecewise constant argument of generalized type. *Nonlinear Anal.: Hybrid Syst.* **2**(2), 456–467 (2008)
3. Akhmet, M.U.: Integral manifolds of differential equations with piecewise constant argument of generalized type. *Nonlinear Anal.* **66**(2), 367–383 (2007)
4. Akhmet, M.U.: Stability of differential equations with piecewise constant arguments of generalized type. *Nonlinear Anal.* **68**(4), 794–803 (2008)
5. Akhmet, M.U., Aruğaslan, D., Yılmaz, E.: Stability analysis of recurrent neural networks with piecewise constant argument of generalized type. *Neural Netw.* **23**(7), 805–811 (2010)
6. Akhmet, M.U., Aruğaslan, D., Yılmaz, E.: Method of Lyapunov functions for differential equations with piecewise constant delay. *J. Comput. Appl. Math.* **235**(16), 4554–4560 (2011)
7. Akhmet, M.U., Büyükdalı, C.: On periodic solutions of differential equations with piecewise constant argument. *Comput. Math. Appl.* **56**(8), 2034–2042 (2008)
8. Aruğaslan, D., Güzel, L.: Stability of the logistic population model with generalized piecewise constant delays. *Adv. Differ. Equ.* **2015**, 1–10 (2015)
9. Assanova, A.T., Uteshova, R.E.: Two-point boundary-value problems for differential equations with generalized piecewise-constant argument. *Ukr. Math. J.* **76**, 705–722 (2024)
10. Bressan, A.: Unique solutions for a class of discontinuous differential equations. *Proc. Amer. Math. Soc.* **104**(3), 772–778 (1988)
11. Aruğaslan-Çinçin, D., Cengiz, N.: Stability analysis of a nonlinear epidemic model with generalized piecewise constant argument. In: Zeynep, S., Gök, A., Aruğaslan-Çinçin, D. (eds) *Emerging Applications of Differential Equations and Game Theory*. pp. 182–208. IGI Global, Hershey PA, USA (2020). <https://doi.org/10.4018/978-1-7998-0134-4.ch009>
12. Busenberg, S., Cooke, K.L.: Models of vertically transmitted diseases with sequential-continuous dynamics. In: Lakshmikantham, V. (ed.) *Nonlinear phenomena in mathematical sciences*, pp. 179–187. Elsevier, USA (1982)
13. Cabada, A., Ferreira, J.B.: First order differential equations with piecewise constant arguments and nonlinear boundary value conditions. *J. Math. Anal. Appl.* **380**(1), 124–136 (2011)
14. Cabada, A., Ferreira, J.B., Nieto, J.J.: Green's function and comparison principles for first order periodic differential equations with piecewise constant arguments. *J. Math. Anal. Appl.* **291**(2), 690–697 (2004)
15. Castillo, S., Pinto, M., Torres, R.: Asymptotic formulae for solutions to impulsive differential equations with piecewise constant argument of generalized type. *Electron. J. Differ. Equ.* **2019**(40), 1–22 (2019)
16. Chiu, K.-S., Pinto, M.: Periodic solutions of differential equations with a general piecewise constant argument and applications. *Electron. J. Qual. Theory Differ. Equ.* **46**, 1–19 (2010)
17. Cooke, K.L., Wiener, J.: A survey of differential equations with piecewise continuous arguments. In: Busenberg, S., Martelli, M. (eds.) *Delay differential equations and dynamical systems*, pp. 1–15. Springer, Berlin, Heidelberg (2006)
18. Cooke, K.L., Wiener, J.: Retarded differential equations with piecewise constant delays. *J. Math. Anal. Appl.* **99**(1), 265–297 (1984)
19. Gordon, R.A.: *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, vol. 4. American Mathematical Soc, Providence (1994)
20. Hájek, O.: Discontinuous differential equations. I. *J. Differ. Equa.* **32**(2), 149–170 (1979)
21. Ivanov, A.F.: Global dynamics of a differential equation with piecewise constant argument. *Nonlinear Anal.* **71**(12), e2384–2389 (2009)
22. Ivanov, A.F., Trofimchuk, S.I.: Periodic solutions of a discretized differential delay equation. *J. Differ. Equa. Appl.* **16**(2–3), 157–171 (2010)
23. Küpper, T., Yuan, R.: On quasi-periodic solutions of differential equations with piecewise constant argument. *J. Math. Anal. Appl.* **267**(1), 173–193 (2002)

24. Lobo, J.Z., Tikare, S., Khuddush, M.: Chaplygin's method for second-order neutral differential equations with piecewise constant deviating arguments. *J. Anal.* **33**, 1021–1039 (2025). <https://doi.org/10.1007/s41478-023-00636-w>
25. Malkin, I.G.: Stability in the case of constantly acting disturbances. *Prikl. Mat.* **8**, 241–245 (1944)
26. Miller, R.K., Michel, A.N.: Ordinary differential equations, Academic Press (1982)
27. Pouso, R.L., Rodriguez-Lopez, J.: Existence and uniqueness of solutions for systems of discontinuous differential equations under localized Bressan-Shen transversality conditions. *J. Math. Anal. Appl.* **492**(1), 124425 (2020)
28. Sakata, S., Hara, T.: Dynamics of a linear differential system with piecewise constant argument. *Dyn. Contin. Discrete Impuls. Syst.* **7**(4), 585–594 (2000)
29. Shah, S.M., Wiener, J.: Advanced differential equations with piecewise constant argument deviations. *Int. J. Math. Math. Sci.* **6**(4), 671–703 (1983)
30. Shevitz, D., Paden, B.: Lyapunov stability theory of nonsmooth systems. *IEEE Trans. Automat. Contr.* **39**(9), 1910–1914 (1994)
31. Sideris, T.C.: Ordinary differential equations and dynamical systems. Atlantis Press, Paris (2013)
32. Topolski, K.: Upper and lower absolutely continuous functions with applications to discontinuous differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2017**(83), 1–12 (2017)
33. Vinter, R.: Optimal control. Birkhäuser Boston, MA (2010)
34. Yoshizawa, T.: Stability theory by Liapunov's second method. Publications of the Mathematical Society of Japan (1966)

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