

On Defining an Admitted Lie Group for First Order Delay Differential Equations with Constant Coefficients

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Abstract—In this paper symmetry analysis is applied to first order delay differential equations with constant coefficients. The determining equations of the admitted Lie group are constructed. This procedure is similar to the way done for ordinary differential equations. The determining equations are then split with respect to the arbitrary elements, and then general solution can be found. The process of splitting the determining equation and solving it is also similar to that done for ordinary differential equations. We obtain the symmetry algebra admitted by this first order delay differential equation with constant coefficients.

Keywords— Delay differential equations, determining equations, group analysis, infinitesimal generator, symmetry.

I. Introduction

Differential equations are widely used in various fields of science and engineering particularly in mathematical modeling. Based on the physical problem we work with, differential equations take various forms like ordinary differential equations, partial differential equations, integro differential equations, functional differential equations, etc. Functional differential equations are further sub classified into retarded, neutral or advanced type, of which delay differential equations are the simplest under the retarded differential equation class. A lot of text on functional differential equations can be found in books, for example, [1-8]. Research on delay differential equations has gained good swing over the last 75 years.

The method of steps is the most commonly known method to solve delay differential equations. Another powerful tool in studying properties of delay differential equations is symmetry analysis. The modern approach of Lie groups is reviewed in [5]. Symmetry analysis along with construction of exact solutions provides a regular procedure for modeling by classifying differential equations with respect to arbitrary elements. The application of symmetry analysis implies construction of the admitted Lie group, which is applied for finding the exact solution.

The non locality due to terms with delay, in a Delay differential equation does not allow the manifold's approach for defining an admitted Lie group (i.e mapping any solution of equations into a solution of the same equation), given by Lie. As a result, the symmetry

approach developed for partial differential equations, cannot be applied to these differential equations. While applying group analysis for constructing the determining equations this is a more appropriate definition of an admitted Lie group: it does not account for the possibilities in [9] where an equation has no solution although it admits a Lie group. Even to study integro differential equations this definition is applied [10-12]. Some retarded equations, like the one in the Cauchy problem, being of the type of integro differential equations, gives us an idea in using the approach, in particular for delay differential equations.

The definition of an admitted Lie group allows us to construct the determining equations. The determining equations are then split with respect to arbitrary elements. The arbitrary elements are parametrical derivatives, In the case of differential equations. In the case of analytical systems the parametrical derivatives are dictated by the Cauchy-Kovalevskaya theorem for a Cauchy type systems and by the Cartan-Kähler theorem for involutive systems. For other types of equations a theorem of existence of the Cauchy problem provides an answer on the question about arbitrary elements.

An approach developed in [13] replaces a delay differential equation by an underdetermined system of differential equations, For this, the classical group analysis is applied. This reduction widens a class of equations admitted by a given Lie group as a result of the underdetermined system. However extending the equations narrows a set of admitted Lie groups.

This paper is devoted to studying symmetry analysis to first order delay differential equations with constant coefficients.

II. Construction of Determining Equations

As delay differential equations have terms with delay (nonlocal terms), this serves as a major hindrance to applying symmetry analysis. To overcome this hindrance, we adopt the following method. We describe the method for a first order delay differential equation,

$$L \equiv x'(t) - F(t, x_t) = 0 \quad (1)$$

Let $\phi(t)$ is a function defined at least on $[t-r, t]$.

Then define a new function $\phi_t: [-r, 0] \rightarrow D$ by

$$\phi_t(s) = \phi(t+s), \quad s \in [-r, 0]$$

where D is an open subset in $\mathbb{R}^n$, J is some interval in $\mathbb{R}$, F is a functional. Here we have used notations accepted in literature given in [3,4,8]. For delay differential equations

$$F(t, x_t) \equiv f(t, \phi(g_1(t)), \dots, \phi(g_m(t))),$$

where $f: [t_0, \beta) \times D^m \rightarrow \mathbb{R}^n$, and $g_j(t) \leq t$ for $t_0 \leq t \leq \beta$ for each $j = 1, 2, \dots, m$. A continuous function $\phi(t), t \in [t_0 - r, t_0 + \beta)$ is called a solution of delay differential equations if it is differentiable in the interval (t_0, β) and satisfies equation (1) in the interval (t_0, β) . We understand $\phi'(t_0)$ as the right hand derivative.

As done for differential equations [14], let the symmetry group G of transformations f_δ :

$$\bar{t} = f_1(t, x; \delta), \quad \bar{x} = f_2(t, x; \delta),$$

depending on a real parameter δ , with $t = f_1(t, x; 0)$ and $x = f_2(t, x; 0)$, map solutions of equation (1) to solutions of the same equation. Instead of a Lie group itself, we usually consider the following infinitesimal generator

$$X = \omega(t, x) \partial_t + Y(t, x) \partial_x$$

where

$$\omega(t, x) = \frac{\partial f_1}{\partial \delta}(t, x; 0), \quad Y(t, x) = \frac{\partial f_2}{\partial \delta}(t, x; 0)$$

Let $x = \phi(t)$ be a solution. A parametrical representation of the transformed function $\phi_\delta(\bar{t})$ is given by the equations $\bar{t} = f_1(t, \phi(t); \delta)$, $\bar{x} = f_2(t, \phi(t); \delta)$. In order to find $\phi_\delta(\bar{t})$, one has to define

$$t = \chi(\bar{t}; \delta), \quad (2)$$

from the equation $\bar{t} = f_1(t, \phi(t); \delta)$. The local inverse function theorem comes to our aid for differential equations. However, for delay differential equation given in (1), one has to define the function χ not only in a neighbourhood of the point t , but also in the interval $[t - r, t]$ and in a right-hand neighbourhood of t . For obtaining this a local inverse function theorem wouldn't be enough. We assume that the given property is possessed by the Lie group. Hence, the function $\phi_\delta(\bar{t})$ is defined by $\phi_\delta(\bar{t}) = f_2(\chi(\bar{t}; \delta), \phi(\chi(\bar{t}; \delta)); \delta)$, and then,

$$\frac{d\phi(\bar{t})}{d\bar{t}} = (f_{1,1} + f_{1,2}\phi'(\chi(\bar{t}; \delta))) \frac{\partial \chi(\bar{t}; \delta)}{\partial \bar{t}}$$

where $f_{1,j}$ means the partial derivative of f with respect to the j^{th} argument. Thus

$$F(\bar{t}, \bar{\phi}_{\bar{t}}) = f(t, f_{11}, \dots, f_{1m})$$

where $f_{1,j} = f_1(\chi(g_j(\bar{t}); \delta), \phi(\chi(g_j(\bar{t}); \delta)); \delta)$, $j = 1, 2, \dots, m$.

Let $\Delta(t, x_t, x') = x' - F(t, x_t)$. Since $\phi(t)$ and $\phi_\delta(\bar{t})$ are solutions, then,

$$\Delta(t, x_t, x') \equiv 0 \quad (3)$$

for $x = \phi(t)$ and $x = \phi_\delta(\bar{t})$. Differentiating the functions $\Delta(t, x_t, x')$, where $x = \phi_\delta(\bar{t})$, with respect to the group parameter δ , then setting $\delta = 0$, we get,

$$\frac{\partial \Delta(t, x_t, x')}{\partial \delta} = 0. \quad (4)$$

The left side of these equations is expressed only through the coefficients of the infinitesimal generator X , their derivatives, the function $\phi(t)$ and its derivatives. This we denote by

$$L(\phi, \omega, Y) \equiv \frac{\partial \Delta(t, x_t, x')}{\partial \delta}, \text{ at } \delta = 0$$

Thus, equation (4) becomes

$$L(\phi, \omega, Y) = 0$$

where $\phi(t)$ is an arbitrary solution of (1).

Note that equation (4) coincides with the equations obtained in the result of an action onto (1) by the prolonged canonical Lie-Bäcklund operator [6] equivalent to the generator X :

$$\hat{X} = (Y(t, x) - \omega(t, x)x') \partial_x + \dots$$

For delay differential equations and in general functional differential equations, the action of the derivative ∂_x has to be considered in the sense of Frechet derivative. It is to be noted that this is the difference in applying the canonical operator as applied for ordinary differential

equations. A Lie group satisfying,

At $[L]$,

$$L(\phi, \omega, Y) = \hat{X}(L) = 0 \quad (5)$$

for any solution of (1), is called an admitted Lie group. Equation (5) are called determining equations.

The notation $[L]$ in (5) means that equation (5) has to be satisfied for any solution of equation (1). (Notation used in [5]).

The definition of admitted Lie group admits the following features (see [15],[16])

1. They must be satisfied for any solution of equation (1).
2. The admitted Lie group does not need to have (2) globally.
3. This definition (In the sense of Lie Bäcklund representation) coincides with the classical definitions of an admitted Lie group in the case of differential equations.
4. Was applied for the Cauchy problem which is a integro-differential equation.
5. Can be applied for finding contact and Lie-Bäcklund transformations and an equivalence group, for delay differential equations and functional differential equations in general.

The determining equations are then split with respect to arbitrary elements. Since arbitrary elements of delay differential equations are contained in determining equations by similar ways, as for differential equations, the process of solving determining equations for delay differential equations is similar to obtaining solutions of determining equations for differential equations.

III. Equivalent Symmetries of the First Order Delay Differential Equation with Constant Coefficients

In this section we extend group analysis to delay differential equations of the form

$$x'(t) = ax(t) + bx(t-r) \quad (6)$$

We define the operator

$$\zeta = (Y - x'\omega) \partial_x + (Y^r - x'^r \omega^r) \partial_{x^r} + Y_{[t]} \partial_{x'} = 0 \quad (7)$$

Let

$$\Delta = x'(t) - ax(t) - bx(t-r) \quad (8)$$

Then by Lie's Invariance condition, at $\Delta = 0, \zeta\Delta = 0$, we get,

$$\begin{aligned} (Y - x'\omega) \frac{\partial}{\partial x} (x'(t) - ax(t) - bx(t-r)) + (Y^r - x'^r \omega^r) \frac{\partial}{\partial x^r} (x'(t) - ax(t) - bx(t-r)) \\ + Y_{[t]} \frac{\partial}{\partial x'} (x'(t) - ax(t) - bx(t-r)) = 0 \end{aligned}$$

Therefore, at (t_0, x_0) , we get,

$$\begin{aligned} \Phi(t_0, x_0, x_1, x_2, x_3) \\ = -aY(t_0, x_0) + ax'(t_0-r)\omega(t_0, x_0) - bY(t_0-r, x_0-r) + bx'(t_0-r)\omega(t_0-r, x_0-r) + Y_{[t]} = 0 \end{aligned}$$

Let, $x'(t_0) = x_1, x_0 = \psi(t_0), x_2 = \psi(t_0-r), x_3 = \psi'(t_0-r)$.

Now,

$$Y_{[t]} = Y_t + x'Y_x - x'\omega_t - x'^2\omega_x$$

$$= Y_t + (ax(t) + bx(t-r))Y_x - (ax(t) + bx(t-r))\omega_t - (ax(t) + bx(t-r))^2\omega_x$$

Therefore, at (t_0, x_0) ,

$$Y_{[t]} = (aY_x(t_0, x_0) - a\omega_t(t_0, x_0)x_0 + (bY_x(t_0, x_0) - b\omega_t(t_0, x_0)x_2 + Y_t(t_0, x_0) - a^2\omega_x(t_0, x_0)x_0^2 - 2ab\omega_x(t_0, x_0)x_0x_2 + b^2\omega_x(t_0, x_0)x_2^2$$

Hence, we get,

$$\Phi(t_0, x_0, x_1, x_2, x_3) = -aY(t_0, x_0) + (a\omega(t_0, x_0) + b\omega(t_0 - r, x_0 - r)x_3 - bY(t_0 - r, x_0 - r) + (aY_x(t_0, x_0) - a\omega_t(t_0, x_0)x_0 + (bY_x(t_0, x_0) - b\omega_t(t_0, x_0)x_2 + Y_t(t_0, x_0) - a^2\omega_x(t_0, x_0)x_0^2 - 2ab\omega_x(t_0, x_0)x_0x_2 + b^2\omega_x(t_0, x_0)x_2^2$$

By splitting this determining equation, we get,

$$aY_x - a\omega_t = 0 \quad (9)$$

$$bY_x - b\omega_t = 0 \quad (10)$$

$$a^2\omega_x = 0 \quad (11)$$

$$2ab\omega_x = 0 \quad (12)$$

$$b^2\omega_x = 0 \quad (13)$$

$$a\omega(t_0, x_0) + b\omega(t_0 - r, x_0 - r) = 0 \quad (14)$$

$$Y_t - bY(t_0 - r, x_0 - r) - aY = 0 \quad (15)$$

From (11), (12) and (13), $\omega_x = 0 \Rightarrow \omega = \omega(t)$

From (9) and (10), $Y(t, x) = \omega'(t)x + \alpha(t)$, where $\alpha(t)$ is an arbitrary function of t . From (14), (15), if $a \neq -b$, $\omega = 0$, hence, $Y = \alpha(t)$, where $\alpha(t)$ is an arbitrary solution of equation (6)

If $a \neq -b$, the generator is given by $X = \alpha(t) \partial_x$, where $\alpha(t)$ is an arbitrary solution of equation (6).

If $a = -b$, from (9-13), $\omega = c$, an arbitrary constant, $Y = \beta(t)$, where $\beta(t)$ is an arbitrary solution of equation (6)

The generator in this case is $X = c \partial_t + \beta(t) \partial_x$, where c is an arbitrary constant and $\beta(t)$ is an arbitrary solution of equation (6).

IV. Conclusions

We have obtained the equivalent symmetries of the first order delay differential equation with constant coefficients. The delay differential equation admits a symmetry algebra of infinite dimension, due to the linear superposition principle.

(1) If the delay differential equation is given by $x'(t) = ax(t) + bx(t-r)$, $a \neq -b$,

then the symmetry algebra is spanned by $X_1 = \alpha(t) \partial_x$.

(2) If the delay differential equation is given by $x'(t) = a(x(t) - x(t-r))$, then the symmetry algebra is spanned by $X_2 = \partial_t + \alpha(t) \partial_x$ and $X_3 = \alpha(t) \partial_x$.

where $\alpha(t)$ is an arbitrary solution of (6).

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