

LIE GROUP ANALYSIS OF THE TIME-DELAYED INVISCID BURGERS' EQUATION

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ABSTRACT. In this paper, we discuss group analysis of first-order delay partial differential equations and use it to obtain symmetries of the Inviscid Burgers' equation with delay, its kernel and extensions of the kernel. We obtain a Lie type invariance condition for first-order delay partial differential equations by using Taylor's theorem for a function of several variables. We obtain the symmetries admitted by this delay partial differential equation. Further, we obtain representations of analytic solutions and the reduced equations from the symmetries.

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1. INTRODUCTION

Delay differential equations are differential equations in which the unknown function appears not only at a given time instant t , but also at earlier instants $t - r, r > 0$. Delay differential equations find a wide range of applications in modeling biological and ecological problems, population dynamics, spread of diseases, control problems, electrical networking, and many more. We can find great detail of literature on delay differential equations in [5].

Transformations leaving objects invariant are called *symmetries*. Symmetries are an important tool in studying laws governing nature. Oliveri in [11] points out that symmetries account for the regularities of the laws that are independent of some inessential circumstances. Reproduction of various experiments at different times and places rely on invariance laws of nature, space translation, rotation, and time translation. The concept of symmetries find a wide range of applications in Applied Mathematics, Physics and Mechanics. In the past, scientists like Newton and Kepler studied the laws of motion as a symmetry principle. One of the most important and interesting application of

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symmetries was observed in 1918 by Nöether while studying the existence of conservation laws in proving a relation connecting symmetries and conservation laws.

Sophus Lie provided the only unified understanding in terms of group analysis as to why differential equations can be solved. However, the presence of delay terms in delay differential equations makes especially the higher order non-linear differential equations difficult to solve. The best way to study the properties of delay differential equations is by group analysis, as there is no analytic method of solving them. We can find a lot of literature on group methods for differential equations without delay in [4]. The motivation to study Lie groups is to model the continuous symmetries of differential equations, in much the same way as finite groups are used in Galois theory to model the discrete symmetries of algebraic equations.

The Burgers' equation which is also known as the Bateman-Burgers' equation is a fundamental partial differential equation which commonly occurs in fluid mechanics, non-linear acoustics, gas dynamics, and traffic flow. For a given field $u(t, x)$ and diffusion coefficient ν , the general form of the Burgers' equation is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}.$$

When the diffusion term is absent, the Burgers' equation becomes the Inviscid Burgers' equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,$$

which is a prototype for conservation laws that can develop discontinuities called *shock waves*.

We can find literature on the Burgers' equation with its applications in [2]. Further, we can find periodic wave shock solutions for the Burgers' equation in [1]. Also, in [13], we can find numerical solutions of the non-homogeneous and non-linear one dimensional Burgers' equation. As differential equations with delay cannot easily be solved, our biggest motivation was to apply symmetry methods in finding a representation of analytic solutions to the Burgers' equation with delay. These solutions obtained will be of utmost importance to researchers in fluid mechanics. The group methods indeed help us in studying the properties of solutions of this delay partial differential equation. The analytic solutions obtained can further be studied to understand their qualitative properties which will help in better modeling of the physical problem at hand.

Before proceeding, we define a Lie group.

Definition 1.1. ([4]) In general, we consider transformations $\bar{t}_i = f_i(t_j, \delta)$, $i, j = 1, 2, \dots, n$ which continuously depend on the parameter δ . Further, we assume that for each i , f_i is a smooth function of the variables t_j and has convergent Taylor series in δ .

The set of transformations is said to form a group if:

- (1) Two transformations carried out in succession are equivalent to another transformation of the set.
- (2) There is a transformation for which the source and image points coincide.
- (3) Each transformation has an inverse.

Remark 1.2. The associative law for groups follows from the *closure* property.

Remark 1.3. In general, the order in which the transformations are carried out matters. If the order does not matter, then we label the group as **abelian**.

Example 1.4. $\bar{t} = a^\delta t, t \in \mathbb{R} \setminus \{0\}$ is a one-parameter group called the *stretching group* is a simple example of a Lie group.

The definition of an admitted Lie group for functional differential equations is proposed in [15] by defining an operator equivalent to the canonical Lie-Bäcklund operator to obtain the determining equations of the functional differential equation. This definition was used by Pue-On [12] to make a complete group classification of second-order delay differential equations. Following a different approach using Taylor's theorem for a function of several variables, a classification for first-order delay differential equations with constant coefficients are made in [7]. Using Taylor's theorem to obtain the determining equations, Lie symmetries of neutral differential equations have been found in [8]. Not many research papers on symmetry analysis of delay partial differential equations are available. In [6], symmetry analysis is used to reduce the number of independent variables of time-fractional partial differential equations and obtain some exact solutions. Lie symmetry theory is extended to the class of space-time fractional differential equations with a delay in [10] and the admitted symmetries of the time fractional Poisson equation with constant delay have been found. Group analysis has been applied to the reaction-diffusion delay differential equation in [9] and the complete group classification of the reaction-diffusion equation with delay is made. New exact explicit solutions of (2+1)-dimensional dispersive long wave equations using similarity transformation method are obtained in [14]. The method therein, reduces the dimension of the partial differential equations by one. Further, very recently in [3] Lie group theoretic method is used to carry out the similarity reduction and solitary

wave solutions of (2+1)-dimensional Date–Jimbo–Kashiwara–Miwa equation. The infinitesimal generators for the governing equation have been obtained therein. These papers add to the motivation for applying symmetry analysis to differential equations with delay as many systems are accurately modeled by delay differential equations and finding analytic solutions to them become of paramount importance. The existing computational technique to obtain symmetries of delay differential equations are based on the Lie–Bäcklund operator and result in magnification in the delay term when obtaining the determining equations. No such magnification appears while using our new technique obtained from Taylor’s theorem for a function of several variables.

In this paper, we perform group analysis of the Inviscid Burgers’ type equation with delay, which is of the form,

$$\frac{\partial u}{\partial t}(t, x) + u(t, x) \frac{\partial u}{\partial x}(t, x) = G(u(t - r, x)), \quad (1.1)$$

where u is a real valued function defined on $I \times D$, and where I is an open interval in $\mathbb{R}$ and D is an open set in $\mathbb{R}$.

Equation (1.1) under study is a non-linear first-order delay partial differential equation and an arbitrary differentiable functional G .

We have used Taylor’s theorem for a function of several variables to obtain a Lie type invariance condition for first-order partial differential equations with delay. The procedure of getting and splitting the determining equations is different from any literature on symmetry analysis of delay partial differential equations. We have studied the Inviscid Burgers’ type equation with delay and an arbitrary differentiable functional and have obtained its symmetries and made a group classification. Further, we have found the kernel and extensions of the kernel to classify (1.1) with respect to its symmetries for an arbitrary and the special case for its functional G . We have obtained a representation of solutions from the invariants and used these representations to reduce the equations to ordinary functional differential equations.

2. LIE TYPE INVARIANCE CONDITION FOR FIRST ORDER DELAY PARTIAL DIFFERENTIAL EQUATIONS

Let $u = u(t, x)$. Then we consider transformations of the form,

$$\bar{t} = f_1(t, x, u; \delta), \quad \bar{x} = f_2(t, x, u; \delta), \quad \bar{u} = f_3(t, x, u; \delta),$$

where f_1, f_2, f_3 are smooth functions of t, x, u having a convergent Taylor series in δ .

Defining,

$$T(t, x, u) = \left. \frac{\partial f_1}{\partial \delta} \right|_{\delta=0}, \quad X(t, x, u) = \left. \frac{\partial f_2}{\partial \delta} \right|_{\delta=0}, \quad U(t, x, u) = \left. \frac{\partial f_3}{\partial \delta} \right|_{\delta=0},$$

we can write the transformations as,

$$\begin{cases} \bar{t} = t + \delta T(t, x, u) + O(\delta^2), \\ \bar{x} = x + \delta X(t, x, u) + O(\delta^2), \\ \bar{u} = u + \delta U(t, x, u) + O(\delta^2). \end{cases}$$

We establish the following Lie type invariance condition for first-order delay partial differential equations using Taylor's theorem for a function of several variables which is new in literature:

Theorem 2.1. Consider the first-order delay partial differential equation

$$F\left(t, t-r, x, u, u(t-r, x), \frac{\partial u}{\partial t}(t, x), \frac{\partial u}{\partial x}(t, x), \frac{\partial u}{\partial x}(t-r, x)\right) = 0, \quad (2.1)$$

where F is defined on a 8-dimensional space $I \times I - r \times D^6$, where D is an open set in $\mathbb{R}$, I is any interval in $\mathbb{R}$, and $I - r = \{y - r : y \in I\}$. Then the Lie type invariance condition is given by

$$TF_t + T^r F_{t^r} + XF_x + UF_u + U^r F_{u^r} + U_{[t]} F_{u_t} + U_{[x]} F_{u_x} + U_{[x]}^r F_{u_x^r} = 0,$$

where $T^r = T(t-r, x, u(t-r, x))$, $U^r = U(t-r, x, u(t-r, x))$, and the total differential operators given by,

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,$$

and,

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots.$$

The extended infinitesimals are given by,

$$U_{[t]} = D_t(U) - u_t D_t(T) - u_x D_t(X),$$

$$U_{[x]} = D_x(U) - u_t D_x(T) - u_x D_x(X),$$

$$U_{[x]}^r = U_x^r + u_x^r U_{u^r}^r - u_{t^r}^r (T_x^r + u_x^r T_{u^r}^r) - u_x^r (X_x^r + u_x^r X_{u^r}^r).$$

Proof. We seek the invariance of equation (2.1) under Lie group of infinitesimal transformations given by,

$$\bar{t} = t + \delta T(t, x, u) + O(\delta^2),$$

$$\bar{x} = x + \delta X(t, x, u) + O(\delta^2),$$

$$\bar{u} = u + \delta U(t, x, u) + O(\delta^2).$$

Then, it naturally follows that,

$$\begin{aligned}\overline{t-r} &= t-r + \delta T(t-r, x, u(t-r, x)) + O(\delta^2), \\ \bar{u}(\overline{t-r}, \bar{x}) &= u(t-r, x) + \delta U(t-r, x, u(t-r, x)) + O(\delta^2).\end{aligned}$$

Let $T(t-r, x, u(t-r, x)) = T^r$, $X(t-r, x, u(t-r, x)) = X^r$,
 $U(t-r, x, u(t-r, x)) = U^r$.

As the partial differential equation given by equation (2.1) contains first-order derivatives $\frac{\partial u}{\partial t}(t, x) = u_t(t, x)$ and $\frac{\partial u}{\partial x}(t, x) = u_x(t, x)$, it is necessary to obtain extended transformations for these.

In analogy with ordinary differential equations, we define the extended transformations $\bar{u}_{\bar{t}}$ and $\bar{u}_{\bar{x}}$ as,

$$\bar{u}_{\bar{t}} = u_t + \delta U_{[t]} + O(\delta^2),$$

$$\bar{u}_{\bar{x}} = u_x + \delta U_{[x]} + O(\delta^2).$$

We introduce the total differential operators D_t and D_x where,

$$\begin{aligned}D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots.\end{aligned}$$

We can write equation (2.1) as $F(t, t-r, x, u, u^r, u_t, u_x, u_x^r) = 0$, where $u^r = u(t-r, x)$.

We shall now construct the extended transformations for u_t and u_x as follows,

$$\begin{aligned}\bar{u}_{\bar{t}} &= \frac{\partial(\bar{u}, \bar{x})}{\partial(\bar{t}, \bar{x})} = \frac{\partial(\bar{u}, \bar{x})}{\partial(t, x)} \bigg/ \frac{\partial(\bar{t}, \bar{x})}{\partial(t, x)} \\ &= \frac{\begin{vmatrix} \bar{u}_t & \bar{u}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix}}{\begin{vmatrix} \bar{t}_t & \bar{t}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix}} \\ &= \frac{\begin{vmatrix} u_t + \delta D_t(U) + O(\delta^2) & u_x + \delta D_x(U) + O(\delta^2) \\ \delta D_t(X) + O(\delta^2) & 1 + \delta D_x(X) + O(\delta^2) \end{vmatrix}}{\begin{vmatrix} 1 + \delta D_t(T) + O(\delta^2) & \delta D_x(T) + O(\delta^2) \\ \delta D_t(X) + O(\delta^2) & 1 + \delta D_x(X) + O(\delta^2) \end{vmatrix}} \\ &= \frac{u_t + \delta[D_t(U) + u_t D_x(X) - u_x D_t(X)] + O(\delta^2)}{1 + \delta[D_t(T) + D_x(X)] + O(\delta^2)} \\ &= u_t + \delta[D_t(U) - u_t D_t(T) - u_x D_t(X)] + O(\delta^2).\end{aligned}$$

Thus we define,

$$U_{[t]} = D_t(U) - u_t D_t(T) - u_x D_t(X),$$

$$U_{[x]} = D_x(U) - u_t D_x(T) - u_x D_x(X).$$

Then,

$$\bar{u}(\overline{t-r}, \bar{x}) = u_x(t-r, x) + \delta U_{[x]}(t-r, x, u(t-r, x)) + O(\delta^2).$$

Now,

$$U_{[t]} = U_t + u_t U_u - u_t(T_t + u_t T_u) - u_x(X_t + u_t X_u), \quad (2.2)$$

and,

$$U_{[x]} = U_x + u_x U_u - u_t(T_x + u_x T_u) - u_x(X_x + u_x X_u). \quad (2.3)$$

$$\begin{aligned} U_{[x]}^r &= U_x(t-r, x, u(t-r, x)) \\ &\quad + u_x(t-r, x)U_{u(t-r, x)}(t-r, x, u(t-r, x, u(t-r, x))) \\ &\quad - u_{t-r}(t-r, x)(T_x(t-r, x, u(t-r, x))) \\ &\quad + u_x(t-r, x)T_{u(t-r, x)}(t-r, x, u(t-r, x)) \\ &\quad - u_x(t-r, x)(X_x(t-r, x, u(t-r, x))) \\ &\quad + u_x X_{u(t-r, x)}(t-r, x, u(t-r, x)). \end{aligned}$$

Then,

$$U_{[x]}^r = U_x^r + u_x^r U_{u^r}^r - u_{t^r}^r(T_x^r + u_x^r T_{u^r}^r) - u_x^r(X_x^r + u_x^r X_{u^r}^r).$$

Then, it follows that,

$$\bar{u}_{\bar{x}}(\overline{t-r}, \bar{x}) = u_x(t-r, x) + \delta U_{[x]}^r + O(\delta^2).$$

For invariance, we need to have,

$$\begin{aligned} 0 &= F(\bar{t}, \overline{t-r}, \bar{x}, \bar{u}, \bar{u}^r, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}}^r) \\ &= F(t + \delta T + O(\delta^2), t-r + \delta T^r + O(\delta^2), x + \delta X + O(\delta^2), \\ &\quad u + \delta U + O(\delta^2), u^r + \delta U^r + O(\delta^2), u_t + \delta U_{[t]} + O(\delta^2), \\ &\quad u_x + \delta U_{[x]} + O(\delta^2), u_x^r + \delta U_{[x]}^r + O(\delta^2)) \\ &= F(t, t-r, x, u, u^r, u_t, u_x, u_x^r) + \delta(TF_t + T^r F_{t^r} + XF_x + UF_u + U^r F_{u^r} \\ &\quad + U_{[t]} F_{u_t} + U_{[x]} F_{u_x} + U_{[x]}^r F_{u_x^r}) + O(\delta^2). \end{aligned}$$

Equating the coefficient of δ , we get,

$$TF_t + T^r F_{t^r} + XF_x + UF_u + U^r F_{u^r} + U_{[t]} F_{u_t} + U_{[x]} F_{u_x} + U_{[x]}^r F_{u_x^r} = 0.$$

□

The infinitesimal generator of the admitted group for the equation given by (2.1) is,

$$\zeta^* = T \frac{\partial}{\partial t} + X \frac{\partial}{\partial x} + U \frac{\partial}{\partial u}.$$

The system of characteristics for this is

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}. \quad (2.4)$$

The first extension is given by,

$$\zeta^{(1)} = T \frac{\partial}{\partial t} + T^r \frac{\partial}{\partial t^r} + X \frac{\partial}{\partial x} + U \frac{\partial}{\partial u} + U^r \frac{\partial}{\partial u^r} + U_{[t]} \frac{\partial}{\partial u_t} + U_{[x]} \frac{\partial}{\partial u_x} + U_{[x]}^r \frac{\partial}{\partial u_x^r}. \quad (2.5)$$

The Lie type invariance condition is given by $\zeta^{(1)} \Delta|_{\Delta=0} = 0$, where $\Delta = F(t, t-r, x, u, u^r, u_t, u_x, u_x^r) = 0$.

3. SYMMETRIES OF THE INVISCID BURGERS' EQUATION WITH DELAY

In this section, we shall obtain symmetries of the Inviscid Burgers' equation with delay given by equation (1.1). We can rewrite equation (1.1) as

$$u_t + uu_x = G(u^r). \text{ Here, } \Delta = u_t + uu_x - G(u^r) = 0.$$

We establish the following result:

Theorem 3.1. The time-delayed Inviscid Burgers' equation given by equation (1.1) admits the three dimensional Lie group generated by

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, \quad \zeta_3^* = \frac{\partial}{\partial x}.$$

Proof. Applying the first extension $\zeta^{(1)}$ to the delay condition $h(t) = t - r$, we get, $T(t, x, u) = T^r$.

Applying the Lie type invariance condition given by equation (2.5) to equation (1.1), we get,

$$u_x U - G'(u^r) U^r + U_{[t]} + u U_{[x]} = 0.$$

Using equations (2.2) and (2.3), we get the determining equation,

$$\begin{aligned} u_x U - G'(u^r) U^r + U_t + u_t U_u - u_t T_t - u_t^2 T_u - u_x X_t - u_x u_t X_u \\ + u(U_x + u_x U_u - u_t T_x - u_x u_t T_u - u_x X_x - u_x^2 X_u) = 0. \end{aligned} \quad (3.1)$$

Since u_x, uu_x, uu_t and u_t are independent variables, we can split equation (3.1) with respect to these variables.

Splitting equation (3.1) with respect to u_x^2 we get $X_u = 0$ which can be solved to give,

$$X(t, x, u) = A(t, x). \quad (3.2)$$

Splitting equation (3.1) with respect to u_t^2 we get $T_u = 0$ which can be solved to give,

$$T(t, x, u) = B(t, x). \quad (3.3)$$

Splitting equation (3.1) with respect to uu_x and solving we get $U_u = A_x(t, x)$ which can be solved to give,

$$U(t, x, u) = A_x(t, x)u + C(t, x). \quad (3.4)$$

Substituting equations (3.2), (3.3) and (3.4) in equation (3.1), we get,

$$\begin{aligned} & 2A_x(t, x)G(u^r) - A_x(t^r, x)u^r G'(u^r) + u^2 A_{xx}(t, x) \\ & + u_x C(t, x) + u C_x(t, x) + u A_{xt}(t, x) - u_x A_t(t, x) + C_t(t, x) - u_t B(t, x) \\ & - uu_t B_x(t, x) - G'(u^r)C(t^r, x) = 0. \end{aligned} \quad (3.5)$$

Equating the coefficient of u^2 to 0, we get $A_{xx}(t, x) = 0$, which can be solved to give,

$$A(t, x) = a_1 tx + b_1(t). \quad (3.6)$$

Equating the coefficient of u to 0, we get,

$$C_x(t, x) + A_{xt}(t, x) - u_t B_x(t, x) = 0. \quad (3.7)$$

Splitting equation (3.7) with respect to u_t , we get, $B_x(t, x) = 0$, which can be solved to give,

$$B = B(t). \quad (3.8)$$

Using equations (3.6) and (3.8), equation (3.7) gives,

$$C(t, x) = -a_1'(t)x + a_3(t). \quad (3.9)$$

Splitting equation (3.5) with respect to u_t , we get,

$$B(t, x) = c_1, \quad (3.10)$$

where c_1 is an arbitrary constant.

Splitting equation (3.5) with respect to u_x , we get,

$$C(t, x) = A_t(t, x). \quad (3.11)$$

Using equations (3.6) and (3.9), equation (3.11) can be solved to give,

$a_1(t) = c_2$, where c_2 is a constant, and $b_1'(t) = a_3(t)$.

Hence equation (3.9) gives,

$$C(t, x) = a_3(t). \quad (3.12)$$

So far we have obtained,

$$A(t, x) = a_1 x + b_1(t), \quad B(t, x) = c_1, \quad C(t, x) = a_3(t).$$

With these values of A, B, C equation (3.5), simplifies to

$$c_2[u^r G'(u^r) - 2G(u^r)] = 0, \quad (3.13)$$

and,

$$a'_3(t) = G'(u^r)a_3(t^r). \quad (3.14)$$

Since equation (3.14) is true for any functional G , we conclude that $a_3(t) = 0$.

Consequently, $b_1(t) = c_3$, where c_3 is an arbitrary constant.

Hence we get the coefficients of the infinitesimal transformation as

$$T(t, x, u) = c_1, \quad X(t, x, u) = c_2x + c_3, \quad U(t, x, u) = c_2u.$$

Thus, the infinitesimal generator of the admitted Lie group is given by

$$\zeta^* = T \frac{\partial}{\partial t} + X \frac{\partial}{\partial x} + U \frac{\partial}{\partial u} \quad (3.15)$$

$$= c_1 \frac{\partial}{\partial t} + (c_2x + c_3) \frac{\partial}{\partial x} + c_2u \frac{\partial}{\partial u}. \quad (3.16)$$

□

4. KERNEL OF THE ADMITTED GENERATORS

Definition 4.1. A *kernel* of admitted generators is the set of symmetries, which are admitted for any functional G appearing in the equation.

This implies that for the Inviscid Burgers' equation with delay (1.1) the coefficients of $G'u^r$ and G vanish. Therefore, $c_2 = 0$. Thus,

$$T(t, x, u) = c_1, \quad X(t, x, u) = c_3, \quad U(t, x, u) = 0.$$

Thus, the obtained infinitesimal generator (which is admitted for any functional G) is given by,

$$\zeta^* = c_1 \frac{\partial}{\partial t} + c_3 \frac{\partial}{\partial x}. \quad (4.1)$$

We have just obtained the following result:

Theorem 4.2. The kernel of the admitted Lie group for the time-delayed Inviscid Burgers' equation (1.1) is two dimensional with generators

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{\partial}{\partial x}.$$

To obtain the symmetry derived by this infinitesimal generator, we need to solve,

$$\frac{d\bar{t}}{d\delta} = T(\bar{t}, \bar{x}, \bar{u}) = c_1, \quad \text{subject to } \bar{t} = t, \quad \text{when } \delta = 0,$$

$$\frac{d\bar{x}}{d\delta} = X(\bar{t}, \bar{x}, \bar{u}) = c_3, \quad \text{subject to } \bar{x} = x, \quad \text{when } \delta = 0,$$

$$\frac{d\bar{u}}{d\delta} = U(\bar{t}, \bar{x}, \bar{u}) = 0, \quad \text{subject to } \bar{u} = u, \quad \text{when } \delta = 0.$$

Solving this, we get,

$$\bar{t} = t + \delta c_1, \quad \bar{x} = x + \delta c_3, \quad \bar{u} = u. \quad (4.2)$$

5. EXTENSIONS OF THE KERNEL

Definition 5.1. Extensions are symmetries for the particular functional G only.

In the case of the Inviscid Burgers' equation with delay (1.1), $\exists G(u^r)$ satisfying equation (3.13).

Here the extensions of the kernel given by (4.1) will be considered.

Since $T(t, x, u) = c_1$, $X(t, x, u) = c_3$, $U(t, x, u) = 0$, are considered in the case of the kernel, the functions for this case are,

$$T(t, x, u) = 0, \quad X(t, x, u) = c_3 x, \quad U(t, x, u) = c_3 u.$$

For the non-trivial case $c_2 \neq 0$, we seek a solution of equation (3.13), which is

$$u^r G'(u^r) = 2G(u^r).$$

This is a separable equation, which can be solved to give,

$$G(u^r) = (u^r)^2 c_4,$$

where c_4 is an arbitrary non-zero constant.

Then the extension of the kernel given by equation (4.1) is

$$\zeta^* = c_2 \left[x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} \right]. \quad (5.1)$$

We have just proved the following result:

Theorem 5.2. The admitted Lie group for the extensions of the kernel of the time-delayed Inviscid Burgers' equation (1.1) is one dimensional with generator

$$\zeta_1^* = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}.$$

To obtain the symmetry derived from equation (5.1) we solve,

$$\frac{d\bar{t}}{d\delta} = T(\bar{t}, \bar{x}, \bar{u}) = 0, \quad \text{subject to } \bar{t} = t, \quad \text{when } \delta = 0,$$

$$\frac{d\bar{x}}{d\delta} = X(\bar{t}, \bar{x}, \bar{u}) = c_2 \bar{x}, \quad \text{subject to } \bar{x} = x, \quad \text{when } \delta = 0,$$

$$\frac{d\bar{u}}{d\delta} = U(\bar{t}, \bar{x}, \bar{u}) = c_2 \bar{u}, \quad \text{subject to } \bar{u} = u, \quad \text{when } \delta = 0.$$

Solving this, we get the symmetries derived from equation (5.1) which are given by,

$$\bar{t} = t, \quad \bar{x} = x e^{c_2 \delta}, \quad \bar{u} = u e^{c_2 \delta}. \quad (5.2)$$

6. REPRESENTATIONS OF ANALYTICAL SOLUTIONS FOR THE TIME-DELAYED INVISCID BURGERS' EQUATION

In this section, we obtain the representation of invariant solutions for equation (1.1) which is given in our following result:

Theorem 6.1. The representation of invariant solutions for the time-delayed Inviscid Burgers' equation given by equation (1.1), for which

- (1) $G(u^r)$ is arbitrary is $u(t, x) = \phi_1(c_3t - c_1x)$.
- (2) $G(u^r) = c_4(u^r)^2$ and for which
 - (a) $c_1 = 0$ is $u(t, x) = (x + c_5)\phi_2(t)$.
 - (b) $c_1 \neq 0$ is $u(t, x) = e^{c_6t}\phi_3((x + c_5)e^{-c_6t})$,

where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Proof. We give a proof as follows:

6.1. Representations of Solutions for

$\mathbf{u}_t(\mathbf{t}, \mathbf{x}) + \mathbf{u}(\mathbf{t}, \mathbf{x})\mathbf{u}_x(\mathbf{t}, \mathbf{x}) = \mathbf{G}(\mathbf{u}(\mathbf{t} - \mathbf{r}, \mathbf{x}))$ for an Arbitrary Functional G .

The system of characteristics for equation (4.1) is given by

$$\frac{dt}{c_1} = \frac{dx}{c_3} = \frac{du}{0}. \quad (6.1)$$

Solving equation (6.1) we get, $u = \text{constant}$ and $c_3t - c_1x = \text{constant}$.

Hence the invariants are u and $c_3t - c_1x$.

For constructing a representation of solutions, the relation between these two invariants is

$$u(t, x) = \phi_1(c_3t - c_1x), \quad (6.2)$$

where ϕ_1 is an arbitrary function.

We call u in equation (6.2) as a *representation of solutions* of equation (1.1) for the infinitesimal generator given by equation (4.1).

At the outset we immediately observe that, if ϕ_1 is bounded, then all solutions $u(t, x)$ are also bounded.

We now discuss the solution for some choices of the function ϕ_1 :

Case i: Let $\phi_1(\cdot) = e^{(\cdot)}$.

Then, $u(t, x) = e^{c_3t - c_1x}$.

We observe that if $c_3 < 0$, then all solutions are positive, decrease with time and eventually tend to zero, as t tends to infinity.

However, if $c_3 > 0$, then all solutions increase exponentially with time and eventually tend to infinity, as t tends to infinity.

Case ii: Let $\phi_1(\cdot) = (\cdot)^m$, for any $m \in \mathbb{N}$.

In this case, all solutions tend to infinity as $t \rightarrow \infty$.

The graphical representations of the analytical solutions for some choices of ϕ_1 are shown in Figures 1 to 3. Here t varies over $[-10, 10]$ and x varies over $[-5, 5]$.

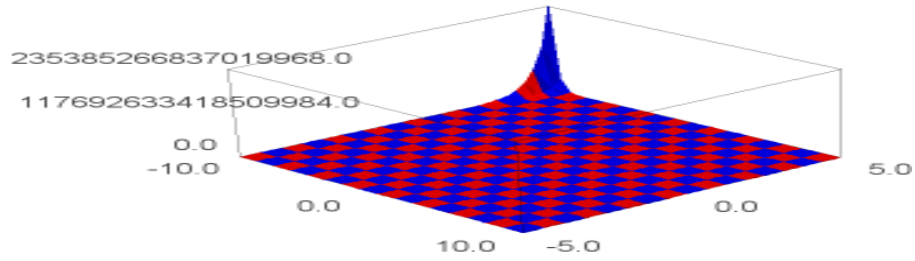


FIGURE 1. $\phi_1(\cdot) = e^{(\cdot)}$, $c_3 = -3$, $c_1 = -2$.

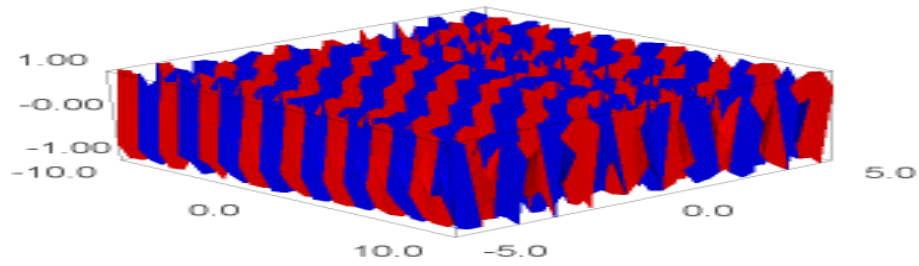


FIGURE 2. $\phi_1(\cdot) = \sin(\cdot)$, $c_3 = 5$, $c_1 = -7$.

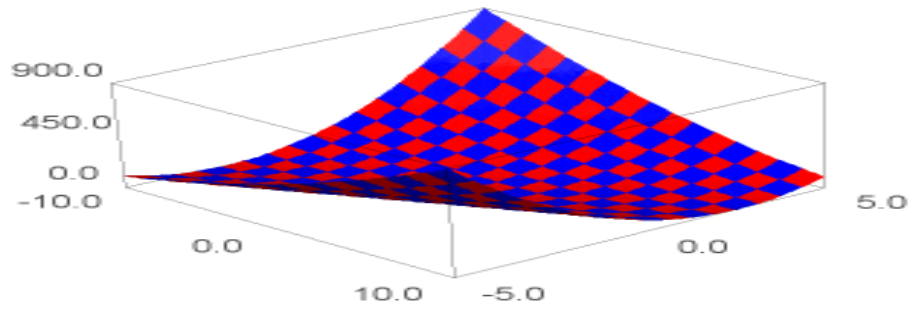


FIGURE 3. $\phi_1(\cdot) = (\cdot)^2$, $c_3 = 1$, $c_1 = 4$.

6.2. Representations of solutions of

$\mathbf{u}_t(\mathbf{t}, \mathbf{x}) + \mathbf{u}(\mathbf{t}, \mathbf{x})\mathbf{u}_x(\mathbf{t}, \mathbf{x}) = \mathbf{G}(\mathbf{u}(\mathbf{t} - \mathbf{r}, \mathbf{x}))$ for $G = c_4(u^r)^2$.

The infinitesimal generator for the equation

$$u_t(t, x) + u(t, x)u_x(t, x) = c_4u^2(t - r, x), \quad (6.3)$$

is a linear combination of the kernel given by equation (4.1) and extension of kernel given by equation (5.1). Thus,

$$\zeta^* = c_1 \frac{\partial}{\partial t} + (c_2x + c_3) \frac{\partial}{\partial x} + c_2u \frac{\partial}{\partial u}. \quad (6.4)$$

The system of characteristics for the infinitesimal generator given by equation (6.4) is

$$\frac{dt}{c_1} = \frac{dx}{c_2x + c_3} = \frac{du}{c_2u}. \quad (6.5)$$

Case 1: Let $c_1 = 0$.

Then solving the equation relating the first and third term of equation (6.5) we get, $t = \text{constant}$.

Solving the equation relating the second and third term of equation (6.5) we get, $\frac{u}{x + c_3/c_2} = \text{constant}$.

In this case, the invariants are t and $\frac{u}{x + c_3/c_2}$.

Since c_2 and c_3 are arbitrary constants and $c_2 \neq 0$, we shall denote $c_5 = \frac{c_3}{c_2}$.

Hence a representation of solutions of equation (6.3) is

$$\frac{u}{x + c_5} = \phi_2(t).$$

That is a representation of solutions for equation (6.3) in this case, is given by

$$u(t, x) = (x + c_5)\phi_2(t), \quad (6.6)$$

where ϕ_2 is an arbitrary function and c_5 is an arbitrary constant.

We immediately observe that if ϕ_2 is a bounded function, then all solutions $u(t, x)$ are bounded if and only if, x belongs to a bounded set.

We now discuss the solution by taking certain choices for the function ϕ_2 :

Case i: Let $\phi_2(t) = e^{\alpha t}$, $\alpha \in \mathbb{R}$.

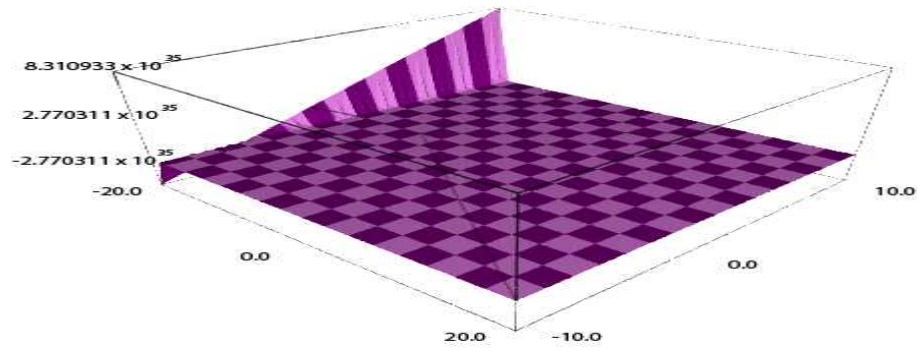
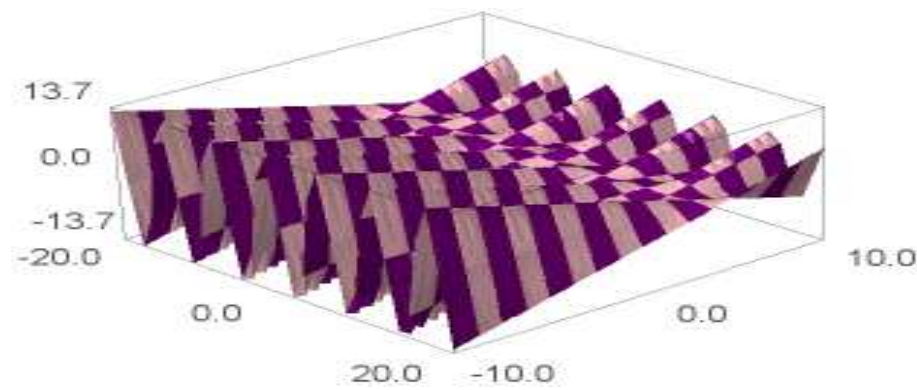
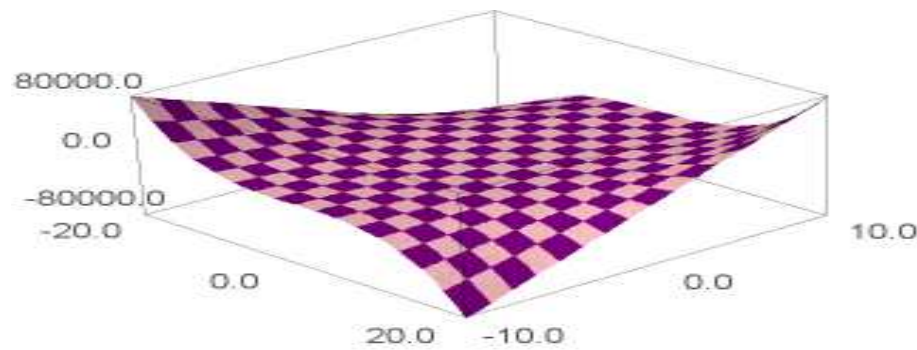
In this case, all solutions $u(t, x)$ tend to 0 as $t \rightarrow \infty$, provided $\alpha < 0$.

On the other hand, all solutions $u(t, x)$ tend to ∞ as $t \rightarrow \infty$, if $\alpha > 0$.

Case ii: Let $\phi_2(t) = t^m$, $m \in \mathbb{N}$.

In this case, all solutions $u(t, x)$ tend to ∞ as $t \rightarrow \infty$.

The graphical representations of the analytical solutions for some choices of ϕ_2 are shown in Figures 4 to 6. Here t varies over $[-20, 20]$ and x varies over $[-10, 10]$.

FIGURE 4. $\phi_2(t) = e^{\alpha t}, \alpha = -4, c_5 = 5$.FIGURE 5. $\phi_2(t) = \sin(t), c_5 = -4$.FIGURE 6. $\phi_2(t) = t^3, c_5 = 0$.

Case 2: Let $c_1 \neq 0$.

Then solving the equation relating the first and second term of equation (6.5)

we get, $(x + c_5)e^{-c_2t/c_1} = \text{constant}$.

Solving the equation relating the first and third term of equation (6.5) we get, $ue^{-c_2t/c_1} = \text{constant}$.

In this case the invariants are $(x + c_5)e^{-c_2t/c_1}$ and ue^{-c_2t/c_1} .

A representation of solutions of equation (6.3) in this case is

$$u = e^{c_2t/c_1} \phi_3 \left((x + c_5)e^{-c_2t/c_1} \right),$$

where ϕ_3 is an arbitrary function. Letting $c_6 = \frac{c_2}{c_1}$, we get a representation of solutions for equation (6.3) as

$$u(t, x) = e^{c_6t} \phi_3 \left((x + c_5)e^{-c_6t} \right). \quad (6.7)$$

We discuss the behavior of the solutions with passing time for some choices of ϕ_3 :

Case i: We first note that if ϕ_3 is bounded and $c_6 < 0$, then all solutions tend to zero, with increase in time, that is, as t tends to infinity.

Case ii: If ϕ_3 is bounded and $c_6 > 0$, then all solutions increase with increase in time, become unbounded and approach infinity as t tends to infinity.

Case iii: If ϕ_3 is an identity function, i.e $\phi_3(.) = (.)$, then $u(t, x) = x + c_5$. In this case, we see that the solution is independent of time, and represents the “steady state” solution.

Case iv: Let $\phi_3(.) = e^{(.)}$.

Then, $u(t, x) = e^{c_6t} e^{(x+c_5)e^{-c_6t}}$.

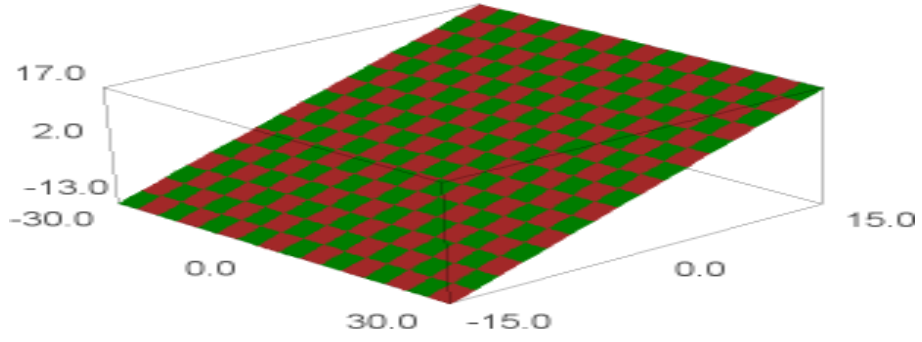
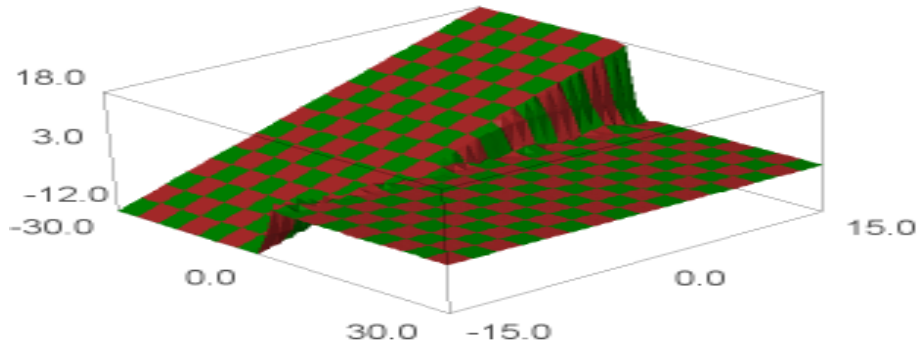
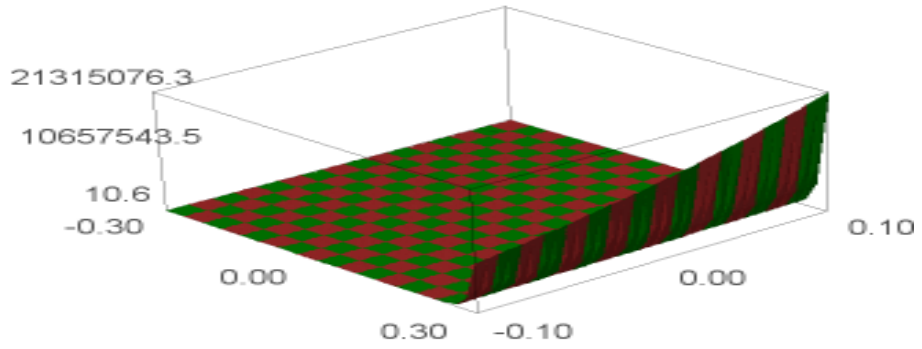
We deduce from here that, if $c_6 > 0$, then all solutions are positive, become unbounded and tend to infinity as time increases to infinity.

However, if $c_6 < 0$, then all solutions exponentially decrease to 0 as $t \rightarrow \infty$, provided $x + c_5$ is non-positive.

The graphical representations of the analytical solutions for some choices of ϕ_3 are shown in Figures 7 to 9. In Figure 7 and Figure 8, t varies over $[-30, 30]$ and x varies over $[-15, 15]$ while in Figure 9, t varies over $[-0.3, 0.3]$ and x varies over $[-0.1, 0.1]$. $\square$

7. REDUCED EQUATIONS

Representations of solutions that we have obtained simplify the Inviscid Burgers' equation with delay (1.1). They reduce the number of independent variables appearing in the equation. The representation when substituted in equation (1.1) reduces it to an ordinary functional differential equation called a *reduced equation*.

FIGURE 7. $\phi_3(\cdot) = (\cdot), c_5 = 2$.FIGURE 8. $\phi_3(\cdot) = \sin(\cdot), c_5 = 3, c_6 = -1$.FIGURE 9. $\phi_3(\cdot) = e^{(\cdot)}, c_5 = 4, c_6 = -5$.

Case 1: Let $u = \phi_1(c_3t - c_1x)$.

Then by substituting this, equation (1.1) becomes

$$c_3\phi_1'(c_3t - c_1x) - c_1\phi_1'(c_3t - c_1x)\phi_1(c_3t - c_1x) = G(\phi_1(c_3(t - r) - c_1x)).$$

Putting $\xi = c_3t - c_1x$ and simplifying the above equation we get,

$$\phi_1'(\xi) = \frac{G(\phi_1(\xi - c_3r))}{c_3 - c_1\phi_1(\xi)}. \quad (7.1)$$

Case 2: Let $u = (x + c_5)\phi_2(t)$.

Then by substituting this, equation (6.3) becomes

$$(x + c_5)\phi_2'(t) + (x + c_5)[\phi_2(t)]^2 = c_4(x + c_5)^2[\phi_2(t - r)]^2,$$

which can be solved to give

$$\phi_2'(t) = c_4(x + c_5)[\phi_2(t - r)]^2 - [\phi_2(t)]^2. \quad (7.2)$$

Case 3: Let $u = e^{c_6t}\phi_3((x + c_5)e^{-c_6t})$.

Then by substituting this and putting $\eta = (x + c_5)e^{-c_6t}$, equation (6.3) becomes

$$-c_6\eta e^{c_6t}\phi_3'(\eta) + c_6e^{c_6t}\phi_3(\eta) + e^{c_6t}\phi_3(\eta)\phi_3'(\eta) = c_4e^{c_6(t-r)}\phi_3\left((x + c_5)e^{-c_6(t-r)}\right),$$

which gives,

$$-c_6\phi_3'(\eta) + c_6\phi_3(\eta) + \phi_3(\eta)\phi_3'(\eta) = c_4e^{-c_6r}\phi_3(\eta e^{c_6r}),$$

which on simplification yields,

$$\phi_3'(\eta) = \frac{c_4e^{-c_6r}\phi_3(\eta e^{c_6r}) - c_6\phi_3(\eta)}{\phi_3(\eta) - \eta c_6}. \quad (7.3)$$

Remark 7.1. It should be noted that equations (7.1), (7.2) and (7.3) are non-linear ordinary functional differential equations.

8. CONCLUSION

The following are our results of symmetry analysis of inviscid Burgers' equation with delay (1.1):

(1): The inviscid Burgers' equation with delay (1.1) admits the three dimensional Lie algebra with generators

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad \zeta_3^* = \frac{\partial}{\partial x}.$$

(2): The kernel of the admitted Lie group is two dimensional with generators

$$\zeta_1^* = \frac{\partial}{\partial t}, \quad \zeta_2^* = \frac{\partial}{\partial x}.$$

(3): The extensions of the kernel of the admitted Lie group is one dimensional with generator

$$\zeta_1^* = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

(4): The symmetry (admitted Lie group) for the inviscid Burgers' equation with delay (1.1) for an arbitrary function G is given by (4.2).

(5): The symmetry (admitted Lie group) for the inviscid Burgers' equation with delay (1.1) for $G = c_4 u^2(t - r, x)$ is given by (5.2).

(6): A representation of solutions for the inviscid Burgers' equation with delay (1.1) for an arbitrary function G is given by (6.2), while a representation of solutions for the inviscid Burgers' equation with delay (1.1) for $G = c_4 u^2(t - r, x)$ is given by (6.6) and (6.7).

(7): The further analysis of these solutions discussed will aid researchers studying Burgers' equation with delay to accurately model and predict the behavior of the system with the passage of time.

(8): The reduced equations of the inviscid Burgers' equation with delay (1.1) are found and given by equations (7.1), (7.2) and (7.3).

9. ACKNOWLEDGEMENT

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