

ERROR ESTIMATES IN PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS FOR A PARTICULAR SECOND ORDER INITIAL VALUE PROBLEM

By

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Abstract

In this paper, we extend the method of successive approximations to second order Initial Value Problems (IVPs) of the type $y'' = f(x, y)$, $y(x_0) = y_0$, $y'(x_0) = y_1$, without converting it to a system of first order differential equations. We obtain an upper bound in the closed form for the difference between two successive iterates. Further, we calculate an error bound for the solution and see that we get a tighter bound for the second order IVP as compared to its first order counterpart.

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1. Introduction

Initial Value Problems (IVPs) are differential equations with some initial conditions that find a wide range of applications in physics and modeling sciences. While every book on ordinary differential equations explains the existence and uniqueness of solutions and methods to compute solutions, it is noteworthy to mention certain developments in finding solutions to IVPs. A new family of explicit schemes for numerical solutions of IVPs are constructed in Ramos et al. [12] by considering suitable rational approximations to the theoretical solution. By varying the step size, Arefin et al. [5] analyzes the solutions of IVPs by discussing Euler's, Modified Euler and Runge-Kutta method. Further, Islam [9] presents Euler method and fourth-order Runge-Kutta method for solving IVPs by investigating and computing errors between the two methods. With the theory for IVP in place, Kumar [10] has illustrated the application of IVPs on a contour integral for Drivastava-Daoust functions of two variables. Having a lot of literature on solving IVPs, it is important to analyze the success of these numerical methods. The popular Euler's method is studied for its accuracy by obtaining the error bounds by Akanbi [4]. With the help of adjoint sensitivity software, Cao and Petzold [6] propose a general method for a posteriori error estimation in the solution of IVPs. By comparing the adjoint method and the classical approach based on the first variational equation, Lang and Verwer[11] address the global error estimation and control for IVPs. Using the preconditioned defect estimates and optimization techniques, Fazal and Neumaier [8] compute the error bounds for approximate solutions of IVPs.

The method of successive approximations by Picard for first order IVPs of the type $y' = f(x, y)$, $y(x_0) = y_0$ found in Agarwal and O'Regan [1], and Deo et al. [7] has been extended to IVPs of the type $y'' = f(x, y)$, $y(x_0) = y_0$, $y'(x_0) = y_1$, without converting these to a system of first order differential equations. The existence of solutions to such second-order IVPs can be found in Schrader [14]. The solutions of such second order IVPs can be computed using the approach in Schrader [13]. Having calculated the first few terms and the difference between successive terms, a pattern is observed for the difference between successive terms which is proved by induction. Finally, an error estimate for solutions has been calculated.

2. Preliminary Results

Consider the first order IVP,

$$y' = f(x, y), y(x_0) = y_0. \quad (2.1)$$

Equation (2.1) is a first order IVP and is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t))dt. \quad (2.2)$$

For this integral equation (2.2), picking $y_0(x) = y_0$, the successive iterates are computed as

$$y_{m+1}(x) = y_0 + \int_{x_0}^x f(t, y_m(t))dt, m = 0, 1, 2, \dots. \quad (2.3)$$

For the above first order IVP we have Picard's local existence theorem as follows:

Theorem 2.1 ([2]). Let us suppose that there is a continuous and bounded function $f(x, y)$ defined on the closed rectangle $\bar{S} : |x - x_0| \leq a, |y - y_0| \leq b$. Let M be the bound of f on $\bar{S}$. Further, let f satisfy the Lipschitz condition on $\bar{S}$ with Lipschitz constant L . If $y_0(x)$ is a continuous function on $\bar{S}$, and if $y(x)$ is the unique solution of the IVP (2.1), then $\{y_m(x)\}$, which is the sequence obtained from Picard's iterative scheme (2.3) converges to $y(x)$. This solution holds true in

$$J_h : |x - x_0| \leq h = \min \left\{ a, \frac{b}{M} \right\}.$$

Theorem 2.2 ([3]). Let the conditions of Theorem 2.1 be satisfied.

Then for all $x \in J_h$ we have the following error estimate:

$$|y(x) - y_m(x)| \leq N \exp(Lh) \min \left\{ 1, \frac{(Lh)^m}{m!} \right\}, m = 0, 1, 2, \dots, \quad (2.4)$$

where $\max_{x \in J_h} |y_1(x) - y_0(x)| \leq N$.

The difference between two successive iterates is bounded and given by

$$|y_m(x) - y_{m-1}(x)| \leq N \frac{(L|x - x_0|)^{m-1}}{(m-1)!}, m = 1, 2, \dots. \quad (2.5)$$

3. Generalizations to second order IVPs of the type $y'' = f(x, y)$, $y(x_0) = y_0$, $y'(x_0) = y_1$

We shall begin this section by establishing a result converting the second order ordinary differential equation to an integral equation.

Theorem 3.1. The solution of the second order IVP

$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y_1, \quad (3.1)$$

is equivalent to the solution of the Volterra integral equation

$$y(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^x (x - t)f(t, y(t))dt. \quad (3.2)$$

Proof. We can rewrite equation (3.1) as $y''(t) = f(t, y(t))$, $y(x_0) = y_0$, $y'(x_0) = y_1$.

Integrating this equation with respect to t between the limits x_0 and x , and using the initial conditions, together with the fundamental theorem of calculus, we get

$$y'(x) = y_1 + \int_{x_0}^x f(t, y(t))dt.$$

Integration of this equation together with the initial conditions, and fundamental theorem of calculus, yields

$$y(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^x \int_{x_0}^{t_2} f(t_1, y(t_1))dt_1 dt_2. \quad (3.3)$$

We will convert the double integral in equation (3.3) to a single integral. To do this, consider

$$I_1(x) = \int_{x_0}^x f(t, y(t))dt, \quad (3.4)$$

and,

$$I_2(x) = \int_{x_0}^x (x - t)f(t, y(t))dt. \quad (3.5)$$

Differentiating equation (3.5) with respect to x by using Leibnitz rule for differentiation under the integral sign, we get

$$\frac{dI_2(x)}{dx} = \int_{x_0}^x \frac{\partial}{\partial x} (x - t)f(t, y(t))dt + (x - x)f(x, y(x))\frac{dx}{dx} - (x - x_0)f(x_0, y(x_0))\frac{dx_0}{dx},$$

which becomes

$$\frac{dI_2(x)}{dx} = \int_{x_0}^x f(t, y(t))dt = I_1(x). \quad (3.6)$$

Differentiating equation (3.6) using Leibnitz rule, and using equation (3.4), we get

$$\frac{d^2 I_2(x)}{dx^2} = f(x, y(x)). \quad (3.7)$$

We note that $I_2(x_0) = 0$ and $I_2'(x_0) = 0$.

We can rewrite equation (3.4) as

$$I_1(x) = \int_{x_0}^x f(t_1, y(t_1))dt_1. \quad (3.8)$$

Using equations (3.6) and (3.8), we have

$$I_2(x) = \int_{x_0}^x I_1(t_2) dt_2 = \int_{x_0}^x \int_{x_0}^{t_2} f(t_1, y(t_1)) dt_1 dt_2. \quad (3.9)$$

From equations (3.5) and (3.9), we see that

$$\int_{x_0}^x \int_{x_0}^{t_2} f(t_1, y(t_1)) dt_1 dt_2 = \int_{x_0}^x (x-t) f(t, y(t)) dt. \quad (3.10)$$

Using equation (3.10) in equation (3.3), we get the desired equation (3.2).

The Picard's iterative scheme for (3.1) is given by

$$y_{m+1}(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^x (x-t)f(t, y_m(t))dt, \quad m = 0, 1, 2, \dots \quad (3.11)$$

Here we generalize Picard's local existence theorem to the second order IVP (3.1).

Theorem 3.2. *Let us suppose that there is a continuous and bounded function $f(x, y)$ defined on the closed rectangle $\bar{S} : |x - x_0| \leq a, |y - y_0| \leq b$. Let M be the bound of f on $\bar{S}$. Further, let f satisfy the Lipschitz condition on $\bar{S}$ with Lipschitz constant L . If $y_0(x)$ is a continuous function on $\bar{S}$, and if $y(x)$ is the unique solution of the IVP (3.1), then $\{y_m(x)\}$, which is the sequence obtained from Picard's iterative scheme (3.11) converges to $y(x)$. This solution holds true in*

$$J_h : |x - x_0| \leq h = \min \left\{ a, \frac{b}{M_1} \right\},$$

where $M_1 = |y_1| + \frac{Ma}{2}$.

Proof. Consider the sequence $\{y_{m+1}(x)\}$, obtained from (3.11). We shall prove that each term of this sequence exists and is continuous in $J_h : |x - x_0| \leq h = \min \left\{ a, \frac{b}{M_1} \right\}$.

By the assumption on $y_0(x)$, the function $F_0(x) = f(x, y_0(x))$ is continuous in J_h , and hence $y_1(x)$ defined as $y_1(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^x (x-t)f(t, y_0(t))dt$ is continuous in J_h .

Further,

$$\begin{aligned} |y_1(x) - y_0| &= |y_0 + (x - x_0)y_1 + \int_{x_0}^x (x-t)f(t, y_0(t))dt - y_0| \\ &\leq |x - x_0| |y_1| + M \int_{x_0}^x |x - t| dt \\ &\leq |y_1| h + \frac{Mh^2}{2} \\ &\leq h \left[|y_1| + \frac{Ma}{2} \right] \\ &\leq b. \end{aligned}$$

Assuming $|y_{m-1}(x) - y_0| \leq b \quad \forall m \geq 2$, it is sufficient to prove that $|y_m(x) - y_0| \leq b$.

The function $F_{m-1}(x) = f(x, y_{m-1}(x))$ is also continuous in J_h as $y_{m-1}(x)$ is continuous there.

Moreover,

$$\begin{aligned} |y_m(x) - y_0| &= |y_0 + (x - x_0)y_1 + \int_{x_0}^x (x-t)f(t, y_{m-1}(t))dt - y_0| \\ &\leq |x - x_0| |y_1| + M \int_{x_0}^x |x - t| dt \\ &\leq |y_1| h + \frac{Mh^2}{2} \\ &\leq h \left[|y_1| + \frac{Ma}{2} \right] \\ &\leq b. \end{aligned}$$

Since $y_0(x)$ and $y_1(x)$ are continuous in J_h , and J_h is closed and bounded, there exists an $N > 0$, such that $\sup_{x \in J_h} |y_1(x) - y_0(x)| \leq N$. We now turn to obtaining a bound for the difference between two successive iterates, which we have found to be

$$|y_{m+1}(x) - y_m(x)| \leq N \frac{L^m |x - x_0|^{2m}}{(2m)!}, \quad m = 0, 1, 2, \dots \quad (3.12)$$

The estimate of bound for difference between successive iterates which is given by (3.12), is now proved via induction.

The case for $m = 0$ is argued earlier.

So we now assume the result to be true for $m = k \geq 2$, i.e., we assume the inductive step as

$$|y_{k+1}(x) - y_k(x)| \leq N \frac{L^k |x - x_0|^{2k}}{(2k)!}, \quad k = 0, 1, 2, \dots$$

We need to prove the result to be true for $m = k + 1$. That is, we need to prove

$$|y_{k+2}(x) - y_{k+1}(x)| \leq N \frac{L^{k+1} |x - x_0|^{2(k+1)}}{(2k+2)!}, \quad k = 0, 1, 2, \dots$$

The main ideas in establishing a bound for the difference between successive iterates begins by substitution of the values of the solutions y_{k+2} and y_{k+1} . Further f is Lipschitz, i.e., $|f(x, y_1) - f(x, y_2)| \leq L |y_1 - y_2|$ in $\bar{S}$. The Lipschitz condition on f then brings us to the induction hypothesis step. The induction is then completed by using the binomial theorem twice.

Now,

$$\begin{aligned} |y_{k+2}(x) - y_{k+1}(x)| &= \left| \left(y_0 + (x - x_0)y_1 + \int_{x_0}^x (x - t)f(t, y_{k+1}(t))dt \right) \right. \\ &\quad \left. - \left(y_0 + (x - x_0)y_1 + \int_{x_0}^x (x - t)f(t, y_k(t))dt \right) \right| \\ &\leq \int_{x_0}^x |x - t| |f(t, y_{k+1}(t)) - f(t, y_k(t))| dt \\ &\leq L \int_{x_0}^x |x - t| |y_{k+1}(t) - y_k(t)| dt \\ &\leq \frac{NL^{k+1}}{(2k)!} \int_{x_0}^x |x - t| |t - x_0|^{2k} dt \\ &= \frac{NL^{k+1}}{(2k)!} \int_{x_0}^x |x - t| \left| t^{2k} - \binom{2k}{1} t^{2k-1} x_0 + \binom{2k}{2} t^{2k-2} x_0^2 - \binom{2k}{3} t^{2k-3} x_0^3 \right. \\ &\quad \left. + \dots + x_0^{2k} \right| dt \\ &= \frac{NL^{k+1}}{(2k)!} \left| \left(\frac{x x_0^{2k+1}}{2k+1} - x x_0^{2k+1} + k x x_0^{2k+1} - \frac{(2k)(2k-1)x x_0^{2k+1}}{6} \right. \right. \\ &\quad \left. - \frac{x_0^{2k+2}}{2k+2} + \frac{2k x_0^{2k+2}}{2k+1} - \frac{(2k-1)x_0^{2k+2}}{2} + \frac{(2k)(2k-2)x_0^{2k+2}}{6} + \dots - \frac{x_0^{2k+2}}{2} \right) \\ &\quad - \left(\frac{x^{2k+2}}{2k+1} - x_0 x^{2k+1} + k x^{2k} x_0^2 - \frac{(2k)(2k-1)x^{2k-1} x_0^3}{6} - \frac{x^{2k+2}}{2k+2} \right. \\ &\quad \left. + \frac{2k x^{2k+1} x_0}{2k+1} - \frac{(2k-1)x^{2k} x_0^2}{2} + \frac{(2k)(2k-2)x^{2k-1} x_0^3}{6} + \dots - \frac{x^2 x_0^{2k}}{2} \right) \Big| \\ &= \frac{NL^{k+1}}{(2k)!} \left(\frac{x^{2k+2} - (2k+2)x^{2k+1} x_0 + \frac{(2k+1)(2k+2)x^{2k} x_0^2}{2} - \dots + x_0^{2k+2}}{(2k+1)(2k+2)} \right) \\ &\leq \frac{NL^{k+1} |x - x_0|^{2k+2}}{(2k)!(2k+1)(2k+2)} \\ &= N \frac{L^{k+1} |x - x_0|^{2(k+1)}}{(2k+2)!}. \end{aligned}$$

By principle of mathematical induction we have established (3.12), $\forall m = 0, 1, 2, \dots, \forall x \in J_h$.

Next,

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{NL^m |x - x_0|^{2m}}{(2m)!} &= N \sum_{m=0}^{\infty} \frac{(\sqrt{L} |x - x_0|)^{2m}}{(2m)!} \\ &\leq N \sum_{m=0}^{\infty} \frac{(\sqrt{L}h)^{2m}}{(2m)!} \\ &= N \exp\left(\frac{Lh}{2}\right) \\ &< \infty. \end{aligned}$$

By Weierstrass M -test, the series $\sum_{m=0}^{\infty} (y_{m+1}(x) - y_m(x))$ converges absolutely and uniformly to a continuous function in interval J_h .

i.e., $y(x) = \lim_{m \rightarrow \infty} y_m(x)$.

This $y(x)$ is a solution of the integral equation (3.2).

The uniqueness of solution easily follows from Gronwall's inequality which can be found in Walter [15].

For if $z(x)$ is another solution in J_h , and $(x, z(x)) \in \bar{S}$, then by using the definitions of $y(x)$ and $z(x)$, we get

$$\begin{aligned}
|y(x) - z(x)| &= \left| \left(y_0 + (x - x_0)y_1 + \int_{x_0}^x (x - t)f(t, y(t))dt \right) \right. \\
&\quad \left. - \left(y_0 + (x - x_0)y_1 + \int_{x_0}^x (x - t)f(t, z(t))dt \right) \right| \\
&\leq \int_{x_0}^x |x - t| \|f(t, y(t)) - f(t, z(t))\| dt \\
&\leq L \int_{x_0}^x |x - t| |y(t) - z(t)| dt \\
&< \epsilon + L \int_{x_0}^x |x - t| |y(t) - z(t)| dt \quad \forall \epsilon > 0 \\
&\leq \epsilon \exp\left(\int_{x_0}^x L |x - t| dt\right).
\end{aligned}$$

The second inequality above is obtained from the assumption in Theorem 3.2 stating that f is Lipschitzian with Lipschitz constant L . The last inequality is obtained using Gronwall's lemma as the previous strict inequality lays the condition for making it favourable to use Gronwall's lemma.

By letting $\epsilon \rightarrow 0$, we get $y(x) = z(x)$, proving uniqueness.

Theorem 3.3. *Let the conditions of Theorem 3.2 be satisfied.*

Then for all $x \in J_h$ we have the following error estimate:

$$|y(x) - y_m(x)| \leq N \exp(\sqrt{L}h) \min \left\{ 1, \frac{(\sqrt{L}h)^{2m}}{(2m)!} \right\}, m = 0, 1, 2, \dots, \quad (3.13)$$

where $\max_{x \in J_h} |y_1(x) - y_0(x)| \leq N$.

Proof. For $n > m$, the triangle inequality gives,

$$\begin{aligned}
|y_n(x) - y_m(x)| &\leq \sum_{k=m}^{n-1} |y_{k+1}(x) - y_k(x)| \\
&\leq \sum_{k=m}^{n-1} NL^k \frac{|x - x_0|^{2k}}{(2k)!} \\
&\leq \sum_{k=m}^{n-1} NL^k \frac{h^{2k}}{(2k)!} \\
&= N \left[\frac{(\sqrt{L}h)^{2m}}{(2m)!} + \frac{(\sqrt{L}h)^{2m+2}}{(2m+2)!} + \frac{(\sqrt{L}h)^{2m+4}}{(2m+4)!} + \dots + \frac{(\sqrt{L}h)^{2n-2}}{(2n-2)!} \right] \\
&\leq N \frac{(\sqrt{L}h)^{2m}}{(2m)!} \left[1 + \frac{\sqrt{L}h}{(2m+1)!} + \frac{(\sqrt{L}h)^2}{(2m+2)!} + \frac{(\sqrt{L}h)^3}{(2m+3)!} + \dots + \frac{(\sqrt{L}h)^{2n-2m-1}}{(2n-1)!} \right] \\
&\leq \frac{N(\sqrt{L}h)^{2m}}{(2m)!} \sum_{k=0}^{\infty} \frac{(\sqrt{L}h)^k}{(k)!} \\
&= \frac{N(\sqrt{L}h)^{2m}}{(2m)!} \exp(\sqrt{L}h).
\end{aligned}$$

We point out that the second inequality in the chain above is obtained from equation (3.12), while the last inequality is due to the fact that every term in the bracket above it is positive. The other inequalities and equalities are mere simplifications.

Letting $n \rightarrow \infty$, and by noting that $\lim_{n \rightarrow \infty} y_n(x) = y(x)$, we get

$$|y(x) - y_m(x)| \leq N \frac{(\sqrt{L}h)^{2m}}{(2m)!} \exp(\sqrt{L}h). \quad (3.14)$$

Further, we already have $|y_n(x) - y_m(x)| \leq N \exp(\sqrt{L}h)$.

Letting $n \rightarrow \infty$, and by noting that $\lim_{n \rightarrow \infty} y_n(x) = y(x)$, we get

$$|y(x) - y_m(x)| \leq N \exp(\sqrt{L}h). \quad (3.15)$$

From equations (3.14) and (3.15), we obtain the desired error bound (3.13).

4. Some Supporting Examples

Let $\bar{S} : |x| \leq \frac{1}{3}, |y - 1| \leq 1$ be a closed rectangle.

Consider the IVP

$$y' = -y, \quad y(0) = 1. \quad (4.1)$$

We know that the exact solution is $y(x) = e^{-x}$. The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= 1 - x, \\ y_2(x) &= 1 - x + \frac{x^2}{2!}, \\ y_3(x) &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 0.000482 + R \quad \text{where} \quad R = \sum_{k=8}^{\infty} \frac{(-1)^k}{3^k k!}. \quad (4.2)$$

For the IVP

$$y'' = -y, \quad y(0) = 1, \quad y'(0) = 1, \quad (4.3)$$

the exact solution is $y(x) = \cos x + \sin x$. The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= 1 + x - \frac{x^2}{2!}, \\ y_2(x) &= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}, \\ y_3(x) &= 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!}. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 0.00000009 + R, \quad \text{where} \quad R = \sum_{k=8}^{\infty} \frac{1}{3^k k!}. \quad (4.4)$$

Comparing equations (4.2) and (4.4), we see that we get a tighter error bound and thus a better approximate for the solution of the second order IVP.

As a final illustration, consider another illustration given below: Let $\bar{S}_1 : |x| \leq \frac{1}{2}, |y - 1| \leq 2$ be a closed rectangle. Consider the IVP

$$y' = e^x + y, \quad y(0) = 1. \quad (4.5)$$

The exact solution is computed to be $y(x) = (x + 1)e^x$. The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= x + e^x, \\ y_2(x) &= \frac{x^2}{2} - 1 + 2e^x, \\ y_3(x) &= \frac{x^3}{3!} - x - 2 + 3e^x. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 0.0060848. \quad (4.6)$$

For the IVP

$$y'' = e^x + y, \quad y(0) = 1, \quad y'(0) = 1, \quad (4.7)$$

the exact solution is found to be

$$y(x) = \frac{1}{4}(2x - 1)e^x + \frac{1}{4}e^{-x} + e^x.$$

The first three iterates by Picard's method are

$$\begin{aligned} y_1(x) &= \frac{x^2}{2!} + e^x, \\ y_2(x) &= \frac{x^4}{4!} - x - 1 + 2e^x, \\ y_3(x) &= \frac{x^6}{6!} - \frac{x^3}{3!} - \frac{x^2}{2!} - 2x - 2 + 3e^x. \end{aligned}$$

The error estimates considering the third iterate is given by

$$|y(x) - y_3(x)| \leq 0.0000018. \quad (4.8)$$

Comparing equations (4.6) and (4.8), we see that we get a tighter error bound and thus a better approximate for the solution of the second order IVP.

5. Conclusion

We have developed Picard's method of successive approximations for a particular type of second order IVP. We have obtained a bound in a closed form for the difference between two successive solutions. Further, we have obtained a tighter error bound on the solution as compared to the existing bound for first order IVPs. We have illustrated our results with suitable examples.

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